Unit III

Properties of Surfaces and Solids

CHAPTER 6

Centroid

KEY CONCEPTS

- Centre of Gravity, Centre of Mass, and Centroid
- Centroid of Curves
- Centroid of Areas
- Centroid of Volumes
- Centroid of Composite Bodies
- Pappus Theorem

6.1 CENTRE OF GRAVITY, CENTRE OF MASS, AND CENTROID

When a body is suspended by a flexible cord through various points, the vertical line passing through the point of suspension passes through a fixed point in each case. This point is called the **centre of gravity** of the body where the entire weight of the body can be assumed to be concentrated. **Centre of mass** is an analogous term where the entire mass of the body can be assumed to be concentrated. Centre of gravity and centre of mass both lie at the same point if the size of the body is very small compared to the size of the earth so that the gravitational field is uniform and parallel at all points of the body. Note that centre of mass is a more basic and fundamental term because it is meaningless to talk about centre of gravity in the absence of a gravitational field.

When the density of a body is uniform, its centre of mass becomes a purely geometric quantity. In such a case, it is referred to as **centroid**. For example, the centre of mass of a sphere with non-uniform density would, in general, not lie at its centre, but its centroid would always be at the centre. In this chapter we would be dealing with centroids only, i.e., bodies of only uniform density would be considered.

The centre of gravity (CG) of a body can be located by applying Varignon's theorem which states that the sum of moments of several forces (which can be considered as the components of the resultant force) about an axis equals the moment of the resultant force about the same axis. In our case, an infinite number of infinitesimal gravitational forces (dW) act on the body, the resultant of which is the weight (W) of the body. If the coordinates of the elemental force and the resultant force are (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$, respectively, then Varignon's theorem for the parallel system of forces gives (coordinate axes being the moment axes)

$$\bar{x}\,W = \int x\,dW;\ \ \bar{y}\,W = \int y\,dW;\ \ \bar{z}\,W = \int z\,dW$$

In a uniform gravitational field, these reduce to (equations of centre of mass)

$$\bar{x} = \frac{\int x\,dm}{m};\ \ \bar{y} = \frac{\int y\,dm}{m};\ \ \bar{z} = \frac{\int z\,dm}{m} \tag{6.1}$$

where *dm* is the elemental mass and *m* is the total mass of the body. The limits of integration should cover the entire body.

6.2 CENTROID OF CURVES

Consider a thin curved wire of length L, uniform cross-sectional area A, and uniform density ρ. Its elemental mass (the mass of the elemental length dL) and the total mass would be $\rho A dL$ and ρAL, respectively. Then, from Eq. 6.1,

$$\bar{x} = \frac{\int x\, dL}{L}; \quad \bar{y} = \frac{\int y\, dL}{L}; \quad \bar{z} = \frac{\int z\, dL}{L} \tag{6.2}$$

where all z-coordinates would be zero for a plane curve lying in the xy-plane. Note that, in general, the centroid of a curve would *not* lie on the curve.

Example 6.1

Derive an expression for the centroid of a thin semicircular arc of mean radius, r.

Solution

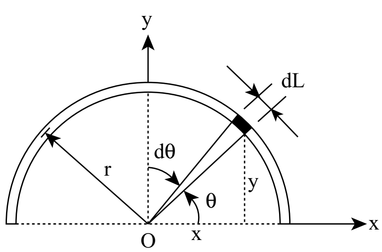

Figure 6.1 Centroid calculation of a semicircular arc

From Fig. 6.1,

$$dL = rd\theta \text{ and } L = \pi r$$

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

From Eq. 6.2,

$$\bar{y} = \frac{\int y\, dL}{L}$$

$$= \frac{\int_0^{\pi} r\sin\theta \cdot rd\theta}{\pi r}$$

$$= \frac{r}{\pi}\int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{r}{\pi}\left[-\cos\theta\right]_0^{\pi} = \frac{r}{\pi}(1+1) = \frac{2r}{\pi}$$

$$\therefore \overline{y} = \frac{2r}{\pi}$$

$$\overline{x} = 0 \text{ (By symmetry)}$$

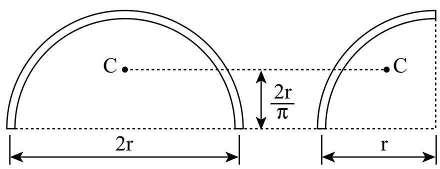

Figure 6.2 Centroid of semicircular and quarter arcs

This is a very important result which one must remember as a formula. Note that *y*-coordinate of the centroid of a quarter circle would also lie at the same level $\left(\overline{y} = \frac{2r}{\pi}\right)$ due to symmetry in left and right halves (Fig. 6.2). One can verify this result by substituting $\frac{\pi r}{2}$ for *L* and integrating between 0 and $\frac{\pi}{2}$. In fact, both $\overline{x}$ and $\overline{y}$ would come out to be the same due to symmetry.

Example 6.2

Derive an expression for the centroid of a thin arc of mean radius *r* and included angle 2α, selecting the symmetrical radial line as *x*-axis.

Solution

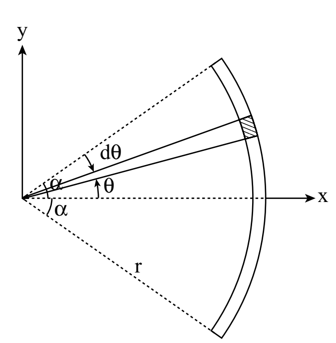

Figure 6.3 Centroid calculation of an arc of radius *r* and included angle 2α

From Fig. 6.3,

$$dL = rd\theta$$

$$L = 2r\alpha$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

From Eq. 6.2,

$$\bar{x} = \frac{\int x\, dL}{L}$$

$$= \frac{\int_{-a}^{a} r^2 \cos\theta\, d\theta}{2ra}$$

$$= \frac{r}{2a}\left[\sin\theta\right]_{-a}^{a}$$

$$= \frac{r \sin a}{a}$$

$$\bar{y} = 0 \text{ (By symmetry)}$$

One can verify that $\bar{x}$ reduces to $\frac{2r}{\pi}$ for $a = \frac{\pi}{2}$, as expected for a semicircular arc.

6.3 CENTROID OF AREAS

A body of small but uniform thickness t can be modelled as a surface area A. Its elemental and total masses would be $\rho t dA$ and $\rho t A$, respectively, where ρ is the mass density (uniform assumed). Then, from Eq. 6.1,

$$\bar{x} = \frac{\int x\, dA}{A}; \quad \bar{y} = \frac{\int y\, dA}{A}; \quad \bar{z} = \frac{\int z\, dA}{A} \tag{6.3}$$

where all z-coordinates would be zero for a plane area lying in the xy-plane. The discussion that follows pertains to plane areas only.

Since integration in one variable is simpler than integration in two (or more) variables, it is better to select a horizontal strip of area $dA = l dy$ (Fig. 6.4a) or a vertical strip of area $dA = l\, dx$ (Fig. 6.4b) instead of a square element of area $dA = dxdy$ (Fig. 6.4c). The thin strips can be approximated by rectangles with their centroids lying at the midpoint of the length l; *x- and y-coordinate of this point are used in Eq. 6.3 for calculating* $\bar{x}$ *and* $\bar{y}$. To cover the entire area, a horizontal strip requires integration between y_{min} and y_{max}. On the other hand, integration is done between x_{min} and x_{max} for a vertical strip (Fig. 6.4).

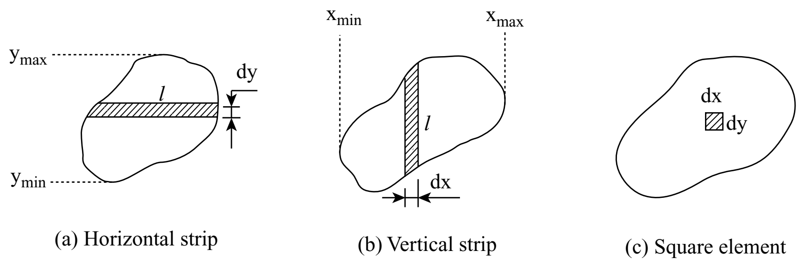

Figure 6.4 Different types of area elements

While both horizontal and vertical strips can be used, there are cases where one of the two would be more convenient than the other. Such a situation arises when the same mathematical expression for the length l does not remain valid in the entire region of integration, due to discontinuity at an intermediate point. As an example, consider the area bounded by curves $y = f(x)$, $y = g(x)$, and the x-axis (Fig. 6.5). With a horizontal strip, the entire area can be covered by a single integration between $y = 0$ and $y = c$ (Fig. 6.5a). On the other hand, two vertical strips of lengths $l_1 = f(x)$ and $l_2 = f(x) - g(x)$ would need to be used because of discontinuity at $x = a$, with the respective limits of integration being $x = 0$ to $x = a$, and $x = a$ to $x = b$ (Fig. 6.5b).

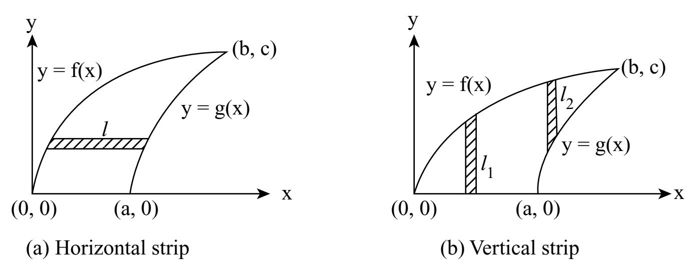

Figure 6.5 Selection between horizontal and vertical strips

Finally, at the risk of repetition, one should note that x, y and z in Eq. 6.3 are the coordinates of the centroid of the chosen area element.

Example 6.3

Determine the distance of the centroid from the base of a triangle of altitude h.

Solution

With a view to avoid double integration, a horizontal or a vertical strip is required to be used as the area element. A vertical strip, however, is not convenient in this case because integration to cover the entire area in one continuous operation would not be possible because of discontinuity at the top-most vertex (Fig. 6.6). Therefore, a horizontal strip, as shown in the figure, has been chosen. For convenience, x-axis is taken to coincide with the base of the triangle.

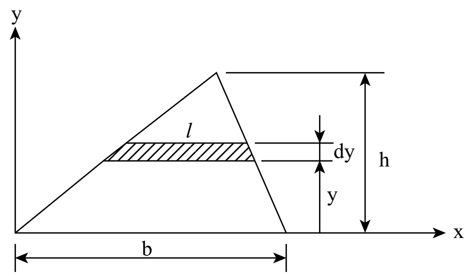

Figure 6.6 Centroid calculation of a triangle

By similar triangles,

$$\frac{l}{b} = \frac{h-y}{h}$$

$$\therefore l = \frac{b(h-y)}{h}$$

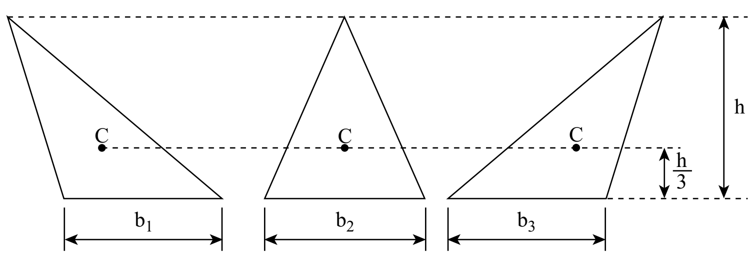

Figure 6.7 Centroid of triangles with the same altitude

From Eq. 6.3,

$$\bar{y}\, A = \int y\, dA$$

$$\bar{y}\left(\frac{bh}{2}\right) = \int_0^h y\, l\, dy$$

$$= \int_0^h \frac{b}{h}(hy - y^2)dy \;\; = \frac{b}{h}\left[h\,\frac{y^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{bh^2}{6}$$

$$\therefore \bar{y} = \frac{h}{3}$$

This is a very important result. One must remember this as a formula (Fig. 6.7).

Calculation of $\bar{x}$ by direct integration in this example is possible but not convenient. A much better approach would be to use the concept of centroid of composite areas in conjunction with the result obtained in this example (*centroid divides the altitudes in the ratio of* 1:2).This is discussed later.

Example 6.4

Locate the centroid of a semicircular disk of radius *r*.

Solution

Method 1 (using horizontal strip)

A horizontal strip is more convenient than a vertical strip (Fig. 6.8).

$$A = \frac{\pi r^2}{2}$$

$$y = r\sin\theta \quad \therefore dy = r\cos\theta\, d\theta$$

$$dA = l\ dy = 2r\cos\theta\, dy = 2r^2 \cos^2\theta\, d\theta$$

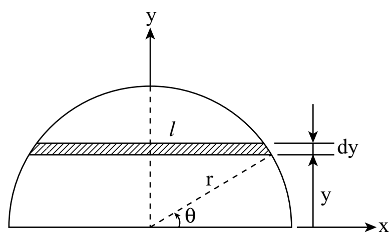

Figure 6.8 Centroid calculation of a semicircular disk using a horizontal strip

From centroid formula,

$$\bar{y}A = \int y\, dA = \int_0^r y\, l\, dy$$

$$= \int_0^{\frac{\pi}{2}} (r\sin\theta)\,(2r^2\cos^2\theta\, d\theta) = 2r^3 \int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\, d\theta$$

Note that the lower and the upper limits of θ correspond to $y = 0$ and $y = r$, respectively.

$$\text{Let } \cos\theta = u \quad \therefore -\sin\theta\, d\theta = du$$

$$\therefore\ \bar{y}A = -2r^3 \int u^2\ du = -2r^3\frac{u^3}{3} = -\frac{2r^3}{3}\cos^3\theta\Big|_0^{\frac{\pi}{2}} = \frac{2r^3}{3}$$

$$\bar{y}\,\frac{\pi r^2}{2} = \frac{2r^3}{3}$$

$$\bar{y} = \frac{4r}{3\pi} \text{ and } \bar{x} = 0 \text{ (By symmetry)}$$

Method 2 (using triangular strip)

A triangular area element can also be used because it covers the entire area when swept between $\theta = 0°$ and $\theta = \pi$, as shown in Fig. 6.9. The shaded area can be approximated by a triangle of base $rd\theta$ and altitude r, the centroid of which would lie at a distance of $r/3$ from the base (see Fig. 6.7). Now,

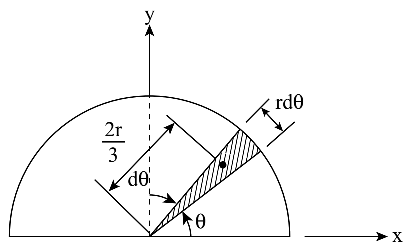

Figure 6.9 Centroid calculation of a semicircular disk using a triangular strip

$$\bar{y}\, A = \int y\, dA$$

$$\bar{y}\,\frac{\pi r^2}{2} = \int_0^{\pi}\left(\frac{2r}{3}\sin\theta\right)\left(\frac{1}{2}r^2\, d\theta\right) = \frac{r^3}{3}\int_0^{\pi}\sin\theta\, d\theta = \frac{r^3}{3}[-\cos\theta]_0^{\pi} = \frac{2r^3}{3}$$

$$\therefore \bar{y} = \frac{4r}{3\pi}$$ which is the same as the previous result.

Recall that in the expression $\int y\, dA$, y is the y-coordinate of the centroid of the area element dA, irrespective of its shape.

Method 3 (using arc element)

A semicircular arc of radius u and thickness du can be chosen as the area element as shown in Fig. 6.10. It covers the entire area when integration is done between $u = 0$ and $u = r$. The centroid of this element lies at a distance of $\frac{2u}{\pi}$ from the diameter (see Ex. 6.1). Now,

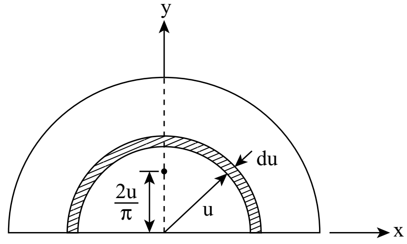

Figure 6.10 Centroid calculation of a semicircular disk using an arc element

$$\bar{y}\ \frac{\pi r^2}{2} = \int y\, dA$$

$$= \int_0^r \left(\frac{2u}{\pi}\right)(\pi\, u\, du) = 2\int_0^r u^2\, du = 2\left[\frac{u^3}{3}\right]_0^r = \frac{2r^3}{3}$$

which gives the same result as obtained earlier.

Method 4 (using vertical strip)

Though not as convenient as the previous methods, a vertical strip can also be used. From Fig. 6.11,

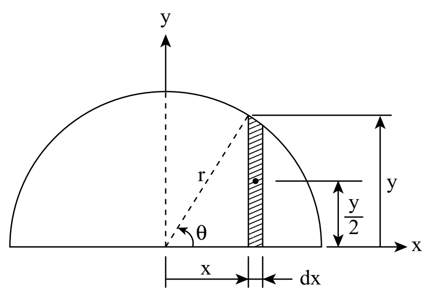

Figure 6.11 Centroid calculation of a semicircular disk using a vertical strip

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx = -r\sin\theta\, d\theta$$
$$dA = ydx$$

$$\bar{y}\ \frac{\pi r^2}{2} = \int \frac{y}{2}\, dA = \frac{1}{2}\int_{-r}^{r} y^2 dx = -\frac{1}{2}\int_{\pi}^{0} r^3 \sin^3\theta\, d\theta$$

$$= \frac{r^3}{2}\int_0^{\pi} \frac{1}{4}(3\sin\theta - \sin 3\theta)\, d\theta \qquad \text{(From trigonometry)}$$

$$= \frac{r^3}{8}\left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi}$$

$$= \frac{r^3}{8}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{r^3}{8}\left[\frac{8}{3} + \frac{8}{3}\right] = \frac{2r^3}{3}$$

which gives the expected result.

Two important points must be carefully noted in this example which can be a source for error:

1. The y-coordinate of the centroid of the element is $\frac{y}{2}$. Therefore, $\int \frac{y}{2}\, dA$ has been evaluated.
2. The lower limit of x is $-r$ which corresponds to $\theta = \pi$. Similarly, the upper limit of x corresponds to $\theta = 0°$. Therefore, integration limits of θ would be π to 0°. Having the limits from 0° to π (which could be a common mistake in this problem) would give an erroneous negative sign after integration.

One can conclude from symmetry arguments that the centroid of a quarter disk also would lie at the same y-level (Fig. 6.12). One must remember this result as a formula.

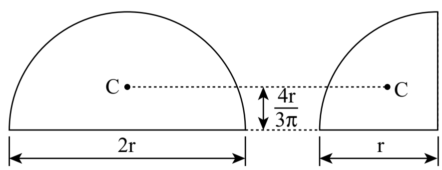

Figure 6.12 Centroid of semicircular and quarter disks

Example 6.5

Locate the centroid of a circular sector of radius r and included angle 2α, selecting the symmetrical radial line as the x-axis.

Solution

Though all the four methods described in Ex. 6.4 can be used, the method involving a triangular strip would be the most convenient. From Fig. 6.13,

$$A = \int dA = \int_{-\alpha}^{\alpha} \frac{1}{2} r^2 d\theta = r^2 \alpha$$

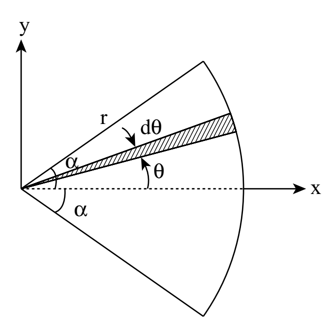

Figure 6.13 Centroid calculation of a circular sector

$$\bar{x}\, A = \int x\, dA$$

$$\bar{x}\, r^2\alpha = \int_{-\alpha}^{\alpha} \left(\frac{2r}{3}\cos\theta\right)\left(\frac{1}{2}r^2 d\theta\right) = \frac{r^3}{3}[\sin\theta]_{-\alpha}^{\alpha} = \frac{2r^3 \sin\alpha}{3}$$

$$\therefore\ \bar{x} = \frac{2r\sin\alpha}{3\alpha} \text{ and } \bar{y} = 0 \quad \text{(By symmetry)}$$

Note that $\bar{x}$ reduces to $\frac{4r}{3\pi}$ for $\alpha = \frac{\pi}{2}$, as expected for a semicircular disk.

Example 6.6

Locate the centroid of the area bounded by lines $x = a$, $y = 0$ and curve $x = \frac{ay^3}{b^3}$.

Solution

$x = a$ and $x = \frac{ay^3}{b^3}$, when solved together, give (a, b) as the point of intersection (Fig. 6.14).

$$A = \int dA = \int_0^a y\, dx = \int_0^a \left(\frac{b^3 x}{a}\right)^{\frac{1}{3}} dx$$

$$= \frac{b}{a^{\frac{1}{3}}}\left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^a = \frac{3b\,a^{\frac{4}{3}}}{4a^{\frac{1}{3}}} = \frac{3ab}{4}$$

$$\bar{x}\, A = \int x\, dA = \int_0^a x\, y\, dx = \int_0^a x\left(\frac{b^3 x}{a}\right)^{\frac{1}{3}} dx$$

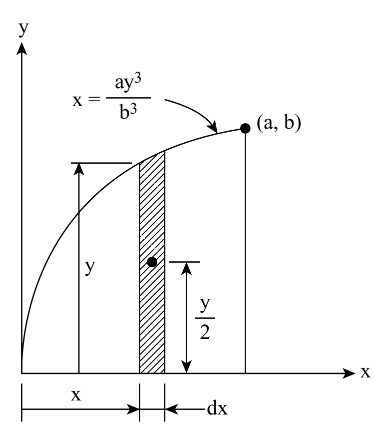

Figure 6.14 Centroid calculation using a vertical strip in Ex. 6.6

$$\bar{x}\,\frac{3ab}{4} = \frac{b}{a^{\frac{1}{3}}}\int_0^a x^{\frac{4}{3}}dx = \frac{3b\,a^{\frac{7}{3}}}{7a^{\frac{1}{3}}} = \frac{3a^2b}{7}$$

$$\therefore\ \bar{x} = \frac{4a}{7}$$

$$\bar{y}\ A = \int \frac{y}{2}\,dA \qquad \left(y\text{-coordinate of the area element is } \frac{y}{2}\right)$$

$$\bar{y}\,\frac{3ab}{4} = \int_0^a \frac{y^2}{2}\,dx = \int_0^a \frac{1}{2}\left(\frac{b^3x}{a}\right)^{\frac{2}{3}} dx = \frac{b^2}{2a^{\frac{2}{3}}}\left[\frac{3a^{\frac{5}{3}}}{5}\right] = \frac{3\,ab^2}{10}$$

$$\therefore\ \bar{y} = \frac{2b}{5}$$

Example 6.7

Locate the centroid of the area bounded by the curve $y = \frac{x^2}{b}$, and lines $y = 2x$ and $y = b$.

Solution

$y = b$ and $y = \frac{x^2}{b}$, when solved together, give (b, b) as the point of intersection (Fig. 6.15). Similarly, the two straight lines intersect at $\left(\frac{b}{2}, b\right)$. Since a vertical strip cannot be integrated in one continuous operation because of discontinuity at $\left(\frac{b}{2}, b\right)$, a horizontal strip as shown in Fig. 6.15 has been taken.

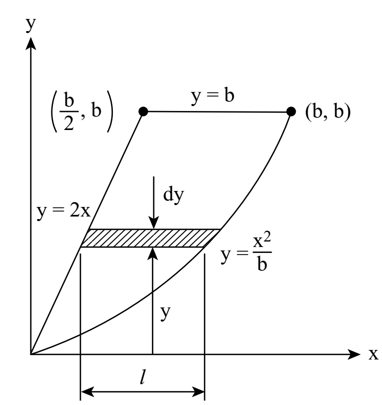

Figure 6.15 Centroid calculation using a horizontal strip in Ex. 6.7

For a given y-coordinate (y), the x-coordinate on the line $y = 2x$ would be $\frac{y}{2}$, and that on the curve $y = \frac{x^2}{b}$ would be $\sqrt{by}$. The length of the horizontal strip shown in Fig. 6.15 would be equal to the difference in these two x-coordinates, i.e., $l = \sqrt{by} - \frac{y}{2}$.

$$A = \int dA = \int_0^b l\,dy = \int_0^b \left(\sqrt{by} - \frac{y}{2} \right) dy = \frac{2b^{\frac{1}{2}} b^{\frac{3}{2}}}{3} - \frac{b^2}{4} = \frac{5b^2}{12}$$

$$\bar{x}\,A = \int \left(\frac{y}{2} + \frac{l}{2} \right) dA \qquad \left(x\text{-coordinate of the area element is } \frac{y}{2} + \frac{l}{2} \right)$$

$$= \int_0^b \frac{1}{2} \left(y + \sqrt{by} - \frac{y}{2} \right) \left(\sqrt{by} - \frac{y}{2} \right) dy = \int_0^b \frac{1}{2} \left(\sqrt{by} + \frac{y}{2} \right) \left(\sqrt{by} - \frac{y}{2} \right) dy$$

$$\bar{x}\,\frac{5b^2}{12} = \int_0^b \frac{1}{2} \left[\left(\sqrt{by} \right)^2 - \left(\frac{y}{2} \right)^2 \right] dy = \frac{1}{2} \left[b\frac{y^2}{2} - \frac{y^3}{12} \right]_0^b = \frac{5b^3}{24}$$

$$\therefore\ \bar{x} = \frac{b}{2}$$

$$\bar{y}\,A = \int y\,dA = \int_0^b y \left(\sqrt{by} - \frac{y}{2} \right) dy = \int_0^b \left(\sqrt{b}\, y^{\frac{3}{2}} - \frac{y^2}{2} \right) dy$$

$$\bar{y}\,\frac{5b^2}{12} = \frac{2\sqrt{b}\ b^{\frac{5}{2}}}{5} - \frac{b^3}{6} = \frac{7b^3}{30}$$

$$\therefore\ \bar{y} = \frac{14b}{25}$$

Example 6.8

Locate the centroid of the area enclosed by curves $y = x^3$ and $y = \sqrt{x}$ in the first quadrant.

Solution

$$A = \int dA = \int_0^1 l\,dy = \int_0^1 (y^{\frac{1}{3}} - y^2)dy = \left[\frac{3y^{\frac{4}{3}}}{4} - \frac{y^3}{3} \right]_0^1 = \frac{5}{12}$$

x-coordinate of the area element $= y^2 + \frac{l}{2} = \frac{1}{2}(y^2 + y^{\frac{1}{3}})$

Notice that this is the average of the x-coordinates on the two curves.

y-coordinate of the area element $= y$

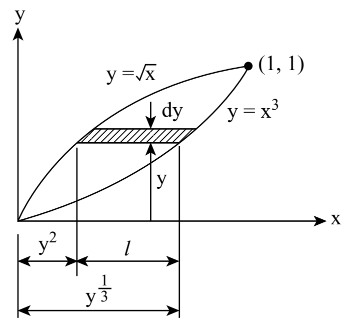

Figure 6.16 Centroid calculation using a horizontal strip in Ex. 6.8

$$\bar{x}\ A = \int \frac{1}{2}\left(y^2 + y^{\frac{1}{3}}\right) dA = \int_0^1 \frac{1}{2}\left(y^2 + y^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}} - y^2\right) dy$$

$$\bar{x}\ \frac{5}{12} = \int_0^1 \frac{1}{2}\left(y^{\frac{2}{3}} - y^4\right) dy = \frac{1}{2}\left[\frac{3y^{\frac{5}{3}}}{5} - \frac{y^5}{5}\right]_0^1 = \frac{1}{5}$$

$$\therefore \bar{x} = \frac{12}{25}$$

$$\bar{y}\ A = \int y\, dA = \int_0^1 y\left(y^{\frac{1}{3}} - y^2\right) dy = \int_0^1 \left(y^{\frac{4}{3}} - y^3\right) dy$$

$$\bar{y}\ \frac{5}{12} = \left[\frac{3y^{\frac{7}{3}}}{7} - \frac{y^4}{4}\right]_0^1 = \frac{5}{28}$$

$$\therefore\ \bar{y} = \frac{3}{7}$$

Example 6.9

Locate the centroid of the area bounded by the two coordinate axes and a circle of radius a with its centre at (a, a), by direct integration.

Solution

Equation of the circle is (Fig. 6.17)

$$(x-a)^2 + (y-a)^2 = a^2$$

$$(y-a)^2 = a^2 - (x-a)^2$$

$$y - a = \pm\sqrt{a^2 - (x-a)^2}$$

$$y = a \pm \sqrt{a^2 - (x-a)^2}$$

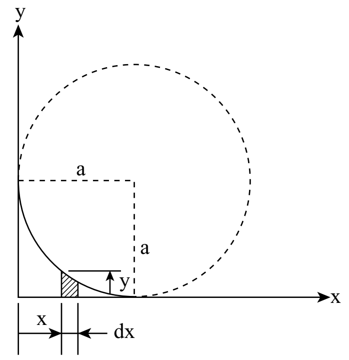

Figure 6.17 Centroid calculation using a vertical strip in Ex. 6.9

This equation indicates that there are two values for y-coordinate, for a given x-coordinate. The lower value corresponds to the distance y shown in the figure. Therefore,

$$y = a - \sqrt{a^2 - (x-a)^2}$$

$$A = \int dA = \int_0^a y\,dx = \int_0^a \left[a - \sqrt{a^2-(x-a)^2}\right] dx = ax\Big|_0^a + \int_a^0 \sqrt{a^2-(x-a)^2}\,dx$$

Let $x - a = a\sin\theta \quad \therefore$ when $x = 0,\ \theta = -\dfrac{\pi}{2}$, and when $x = a,\ \theta = 0°$

$$dx = a\cos\theta\,d\theta$$

$$\therefore\ A = a^2 + \int_0^{-\frac{\pi}{2}} a^2\cos^2\theta\,d\theta = a^2 + \frac{a^2}{2}\int_0^{-\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta = a^2 + \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{-\frac{\pi}{2}}$$

$$= a^2 + \frac{a^2}{2}\left(-\frac{\pi}{2}\right) = a^2 - \frac{\pi a^2}{4} = a^2\left(1-\frac{\pi}{4}\right) = a^2\left(\frac{4-\pi}{4}\right)$$

Note that we could have easily obtained this result by subtracting the area of the quarter circle from the area of the square of side a, in Fig. 6.17.

x-coordinate of the area element $= x$

$$\therefore\ \bar{x}\,A = \int x\,dA = \int_0^a x\left[a - \sqrt{a^2-(x-a)^2}\right] dx$$

$$= \int_0^a x(a - a\cos\theta)\,dx \ \ \text{(Using the previous substitution)}$$

$$= a\left[\frac{x^2}{2}\right]_0^a - \int_{-\frac{\pi}{2}}^0 a^3(1+\sin\theta)\cos^2\theta\,d\theta$$

$$= \frac{a^3}{2} - a^3 \int_{-\frac{\pi}{2}}^{0} \left[\frac{1}{2}(1 + \cos 2\theta) + \sin\theta \ \cos^2\theta \right] d\theta$$

$$= \frac{a^3}{2} - a^3 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{-\frac{\pi}{2}}^{0} - a^3 \int_{-\frac{\pi}{2}}^{0} \sin\theta \cos^2\theta \ d\theta$$

$$\left[\begin{array}{l} \text{Let } \cos\theta = u \quad \therefore -\sin\theta \ d\theta = du \\ \therefore \int \sin\theta \cos^2\theta \ d\theta = -\int u^2 \ du = -\frac{u^3}{3} = -\frac{\cos^3\theta}{3} \end{array} \right]$$

$$= \frac{a^3}{2} - a^3 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\cos^3\theta}{3} \right]_{-\frac{\pi}{2}}^{0}$$

$$\bar{x}\, a^2 \left(\frac{4-\pi}{4} \right) = \frac{a^3}{2} - a^3 \left[-\frac{1}{3} - \left(-\frac{\pi}{4} \right) \right] = a^3 \left(\frac{5}{6} - \frac{\pi}{4} \right)$$

$$\bar{x} \left(\frac{4-\pi}{4} \right) = a \left(\frac{10 - 3\pi}{12} \right)$$

$$\therefore \ \bar{x} = \frac{(10 - 3\pi)a}{3(4-\pi)} = 0.2234 \ a = \bar{y} \text{ (By symmetry)}$$

Example 6.10

Find the centroid of the area under the half sine curve shown in Fig. 6.18.

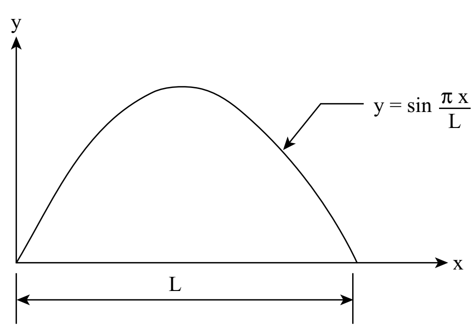

Figure 6.18 Figure for Ex. 6.10

Solution

$$A = \int_0^L y\,dx = \int_0^L \sin\frac{\pi x}{L}\,dx$$

$$= -\frac{L}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = -\frac{L}{\pi}(-1-1) = \frac{2L}{\pi}$$

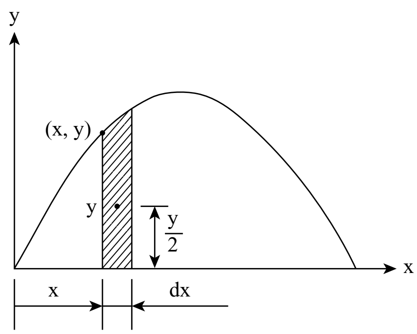

Figure 6.19 Centroid calculation using a vertical strip in Ex. 6.10

$$\bar{y}\,A = \int y\,dA = \int_0^L \frac{y}{2}(y\,dx) = \int_0^L \frac{1}{2}\sin^2\frac{\pi x}{L}\,dx$$

$$= \frac{1}{4}\int_0^L \left(1-\cos\frac{2\pi x}{L}\right)dx = \frac{1}{4}\left[x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_0^L = \frac{L}{4}$$

$$\bar{y} = \frac{L}{4}\times\frac{\pi}{2L} = \frac{\pi}{8}$$

By symmetry, $\bar{x} = \dfrac{L}{2}$

Recall that, in the formula $\bar{y}\,A = \int y\,dA$, y is the y-coordinate of the centroid of the elemental area, which is $\dfrac{y}{2}$ in the present case.

6.4 CENTROID OF VOLUMES

For a body of uniform density, Eq. 6.1 reduces to

$$\bar{x} = \frac{\int x\, dV}{V};\ \ \bar{y} = \frac{\int y\, dV}{V};\ \ \bar{z} = \frac{\int z\, dV}{V} \tag{6.4}$$

where dV is the elemental volume and V is the total volume. Further, x, y and z are the coordinates of the *centroid of the chosen volume element*.

Example 6.11

Locate the centroid of a hemisphere of radius r with respect to its base.

Solution

$$x = r\cos\theta \quad \therefore\ dx = -r\sin\theta\, d\theta$$
$$y = r\sin\theta$$

The chosen volume element is a disk of radius y. Therefore,

$$dV = \pi y^2\, dx$$

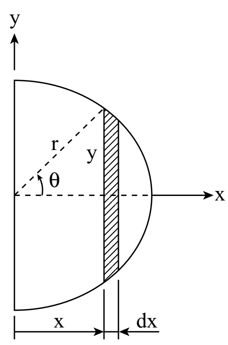

Figure 6.20 Centroid calculation using a disk element in Ex. 6.11

$$\int x\, dV = \int_0^r \pi\, x\, y^2\, dx$$
$$= \int_{\frac{\pi}{2}}^{0} \pi\,(r\cos\theta)\,(r^2\sin^2\theta)(-r\sin\theta\, d\theta)$$

$$\left[\begin{array}{l}\text{Integration limits: When } x = 0, \theta = \dfrac{\pi}{2}, \text{ and} \\ \text{when } x = r, \theta = 0°, \text{ being the lower and the upper limits, respectively.}\end{array}\right]$$

$$= \int_0^{\frac{\pi}{2}} \pi\, r^4 \sin^3\theta \cos\theta\, d\theta$$

$$\text{Let } \sin\theta = u \quad \therefore \cos\theta\, d\theta = du$$

$$\therefore \int x\, dV = \int \pi\, r^4 u^3 du = \pi r^4 \frac{u^4}{4} = \left.\frac{\pi r^4 \sin^4\theta}{4}\right|_0^{\frac{\pi}{2}} = \frac{\pi r^4}{4}$$

For a hemisphere, $V = \dfrac{2}{3}\pi r^3$ (One can evaluate $\int dV$ to obtain the same result).

Therefore, from Eq. 6.4,

$$\bar{x} = \frac{\int x\, dV}{V} = \frac{\pi r^4}{4} \cdot \frac{3}{2\pi r^3} = \frac{3r}{8}$$

$$\text{From symmetry, } \bar{y} = \bar{z} = 0$$

Note that the selection of a disk element reduces triple integration to integration in a single variable. This technique should be used wherever possible.

Example 6.12

Locate the centroid of a cone of base diameter b and height h.

Solution

Consider a disk element of radius r and thickness dy. Therefore,

$$dV = \pi\, r^2 dy$$

From similar triangles,

$$\frac{2r}{b} = \frac{h-y}{h} \quad \therefore r = \frac{b}{2h}(h-y)$$

$$\therefore \int y\, dV = \int_0^h \pi\, r^2 y\, dy$$

$$= \int_0^h \pi \frac{b^2}{4h^2}(h^2 - 2hy + y^2)y\,dy$$

$$= \frac{\pi b^2}{4h^2}\int_0^h (h^2 y - 2hy^2 + y^3)\,dy$$

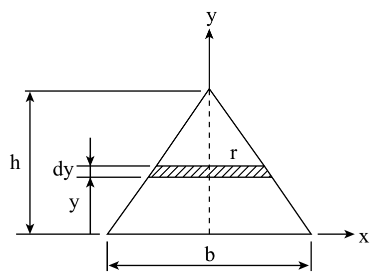

Figure 6.21 Centroid calculation using a disk element in Ex. 6.12

$$= \frac{\pi b^2}{4h^2}\left[h^2\frac{y^2}{2} - 2h\frac{y^3}{3} + \frac{y^4}{4}\right]_0^h$$

$$= \frac{\pi b^2 h^2}{4}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = \frac{\pi b^2 h^2}{48}$$

Volume of the cone, $V = \frac{1}{3}\pi\left(\frac{b}{2}\right)^2 h = \frac{\pi b^2 h}{12}$

(Integration $\int dV$ would give the same result.)

Now, from Eq. 6.4,

$$\bar{y} = \frac{\int y\,dV}{V} = \frac{\pi b^2 h^2}{48}\cdot\frac{12}{\pi b^2 h} = \frac{h}{4}$$

From symmetry, $\bar{x} = \bar{z} = 0$

Example 6.13

A solid is formed by revolving the shaded area by 360° about the y-axis as shown in Fig. 6.22. Its maximum radius and height are b and a, respectively. The parabolic boundary of this solid in the xy-plane is of the form $y = kx^2$. Locate its centroid.

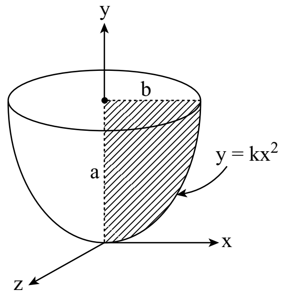

Figure 6.22 Figure for Ex. 6.13

Solution

The curve $y = kx^2$ passes through (b, a).

Therefore,

$$a = kb^2$$

$$k = \frac{a}{b^2}$$

Thus, the equation of the curve is

$$y = \frac{a}{b^2}x^2$$

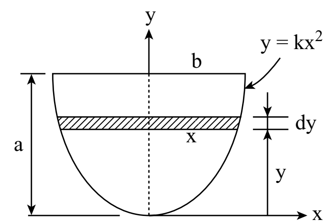

Figure 6.23 Centroid calculation using a disk element in Ex. 6.13

Volume of the disc element,

$$dV = \pi x^2 \, dy$$

Therefore,

$$\int y\,dV = \int_0^a \pi x^2 y\,dy = \int_0^a \pi \frac{b^2 y}{a} y\,dy = \frac{\pi b^2}{a}\left[\frac{y^3}{3}\right]_0^a = \frac{\pi b^2 a^2}{3}$$

$$V = \int dV = \int_0^a \pi x^2 dy = \int_0^a \pi \frac{b^2 y}{a} dy = \frac{\pi b^2}{a}\left[\frac{y^2}{2}\right]_0^a = \frac{\pi b^2 a}{2}$$

$$\bar{y} = \frac{\int y dV}{V} = \frac{\pi b^2 a^2}{3} \cdot \frac{2}{\pi b^2 a} = \frac{2a}{3}$$

By symmetry, $\bar{x} = \bar{z} = 0$

Example 6.14

Locate the centroid of the frustrum of the cone shown in Fig. 6.24.

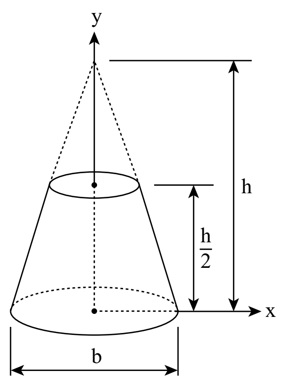

Figure 6.24 Figure for Ex. 6.14

Solution

From similar triangles,

$$\frac{2r}{b} = \frac{h-y}{h} \quad \therefore r = \frac{b}{2h}(h-y)$$

Elemental volume, $dV = \pi r^2 dy = \dfrac{\pi b^2}{4h^2}(h-y)^2\,dy$

Total volume, $V = \int dV = \dfrac{\pi b^2}{4h^2}\displaystyle\int_0^{\frac{h}{2}} (h-y)^2\,dy$

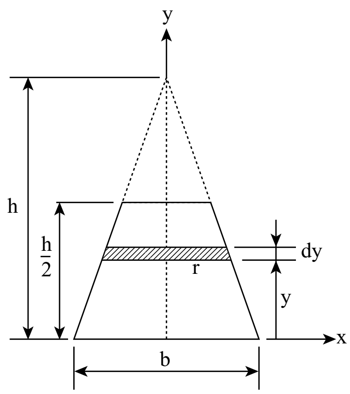

Figure 6.25 Centroid calculation using a disk element in Ex. 6.14

$$= \frac{\pi b^2}{4h^2}\left[\frac{(h-y)^3}{-3}\right]_0^{\frac{h}{2}}$$

$$= \frac{\pi b^2}{12h^2}\left(h^3 - \frac{h^3}{8}\right) = \frac{7\pi b^2 h}{96}$$

Integration could have been avoided by using the formula for the volume of a cone.

From similar triangles, diameter at $y = \frac{h}{2}$ would be $\frac{b}{2}$.

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{b}{2}\right)^2 h - \frac{1}{3}\pi\left(\frac{b}{4}\right)^2\left(\frac{h}{2}\right)$$

$$= \frac{1}{3}\pi b^2 h\left(\frac{1}{4} - \frac{1}{32}\right) = \frac{7\pi b^2 h}{96}$$ which is the same as before.

Now, $$\int y\,dV = \frac{\pi b^2}{4h^2}\int_0^{\frac{h}{2}} (h^2 - 2hy + y^2)\,y\,dy$$

$$= \frac{\pi b^2}{4h^2}\left[h^2\frac{y^2}{2} - 2h\frac{y^3}{3} + \frac{y^4}{4}\right]_0^{\frac{h}{2}}$$

$$= \frac{\pi b^2 h^2}{4}\left(\frac{1}{8} - \frac{1}{12} + \frac{1}{64}\right) = \frac{11\pi b^2 h^2}{768}$$

$$\bar{y} = \frac{\int y\,dV}{V} = \frac{11\pi b^2 h^2}{768}\cdot\frac{96}{7\pi b^2 h} = \frac{11h}{56}$$

By symmetry, $\bar{x} = \bar{z} = 0$

Example 6.15

A solid, half of a truncated cone, is formed by revolving the shaded area shown in Fig. 6.26 by 180° about the x-axis. Determine the y-coordinate of the centroid.

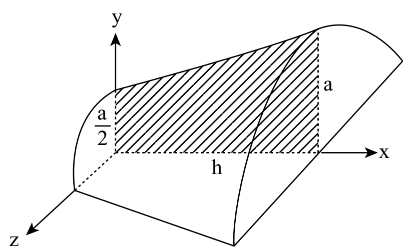

Figure 6.26 Figure for Ex. 6.15

Solution

From similar triangles,

$$\frac{y-\frac{a}{2}}{a-\frac{a}{2}}=\frac{x}{h}$$

$$y=\frac{a}{2}+\left(\frac{a}{2h}\right)x$$

$$dy=\frac{a}{2h}dx$$

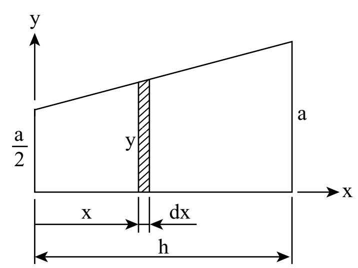

Figure 6.27 Centroid calculation using a semicircular disk element in Ex. 6.15

$$\text{Elemental volume, } dV=\frac{\pi y^2}{2}dx$$

$$\text{Total volume, } V=\int dV=\int_0^h \frac{\pi y^2}{2}dx=\int_{\frac{a}{2}}^{a}\frac{\pi y^2}{2}\cdot\frac{2h}{a}dy$$

$$\left[\text{Integration limits: When } x = 0,\ y = \frac{a}{2}, \text{ and when } x = h,\ y = a\right]$$

$$= \frac{\pi h}{a}\left[\frac{y^3}{3}\right]_{\frac{a}{2}}^{a} = \frac{\pi h}{a}\left(\frac{a^3}{3} - \frac{a^3}{24}\right) = \frac{7\pi h a^2}{24}$$

We will use the result that the centroid of a semicircular disk of radius r lies at a height of $\frac{4r}{3\pi}$ from its base (see Fig. 6.12).

Therefore,

$$\int y\,dV = \int_0^h \frac{4y}{3\pi}\cdot\frac{\pi y^2}{2}\,dx = \int_{\frac{a}{2}}^{a} \frac{2y^3}{3}\cdot\frac{2h}{a}\,dy$$

$$= \frac{4h}{3a}\left[\frac{y^4}{4}\right]_{\frac{a}{2}}^{a} = \frac{h}{3a}\left(a^4 - \frac{a^4}{16}\right) = \frac{5ha^3}{16}$$

$$\therefore\quad \bar{y} = \frac{\int y\,dV}{V} = \frac{5ha^3}{16}\cdot\frac{24}{7\pi h a^2} = \frac{15a}{14\pi}$$

6.5 CENTROID OF COMPOSITE BODIES

The principle which was used to derive Eq. 6.1 can be extended also to a body composed of finite shapes with *known* centroidal positions of the individual parts. Thus, if the individual masses are m_1, m_2, m_3…. with the respective coordinates of the centroids being $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$, $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$, $(\bar{x}_3, \bar{y}_3, \bar{z}_3)$..., then Eq. 6.1 would take the form

$$\bar{x} = \frac{m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i\bar{x}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i\bar{x}_i}{m}$$

$$\bar{y} = \frac{m_1\bar{y}_1 + m_2\bar{y}_2 + m_3\bar{y}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i\bar{y}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i\bar{y}_i}{m}$$

$$\bar{z} = \frac{m_1\bar{z}_1 + m_2\bar{z}_2 + m_3\bar{z}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_{i=1}^{n} m_i\bar{z}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i\bar{z}_i}{m}$$

Analogous relations hold for composite lines, areas and volumes, where the *m*'s are replaced by *L*'s, *A*'s and *V*'s, respectively.

Composite lines

$$\bar{x} = \frac{\sum_{i=1}^{n} L_i \, \bar{x}_i}{L}; \quad \bar{y} = \frac{\sum_{i=1}^{n} L_i \, \bar{y}_i}{L}; \quad \bar{z} = \frac{\sum_{i=1}^{n} L_i \, \bar{z}_i}{L} \tag{6.5}$$

Composite areas

$$\bar{x} = \frac{\sum_{i=1}^{n} A_i \, \bar{x}_i}{A}; \quad \bar{y} = \frac{\sum_{i=1}^{n} A_i \, \bar{y}_i}{A}; \quad \bar{z} = \frac{\sum_{i=1}^{n} A_i \, \bar{z}_i}{A} \tag{6.6}$$

Composite volumes

$$\bar{x} = \frac{\sum_{i=1}^{n} V_i \, \bar{x}_i}{V}; \quad \bar{y} = \frac{\sum_{i=1}^{n} V_i \, \bar{y}_i}{V}; \quad \bar{z} = \frac{\sum_{i=1}^{n} V_i \, \bar{z}_i}{V} \tag{6.7}$$

If a hole or cavity is considered one of the constituent parts of a composite body, the corresponding mass of the hole or cavity would be a negative quantity. The concept of negative mass/area/volume is often very convenient in defining the shape of a complex composite body, resulting in simple calculations (see Ex. 6.20).

If a composite body consists of more than two parts, Eqs. 6.5–6.7 should preferably be used in a tabular form to minimise the chances of calculation mistakes. Any coordinate system can be used for measuring centroidal distances in these equations. The centroidal distances would be positive or negative, depending on whether these lie on the positive side or the negative side of the corresponding axes.

Example 6.16

A uniform rod is bent into the shape shown in Fig. 6.28. It is pivoted at O. If the rod remains in equilibrium in the position shown (the straight part being horizontal), find the relation between the length *a* and the mean radius *r*.

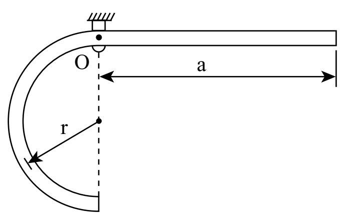

Figure 6.28 Figure for Ex. 6.16

Solution

Equilibrium in the position shown would be possible if the centroid of the bent rod lies on the vertical line passing through the pivot O. Centroid of the circular part lies at a distance of $\frac{2r}{\pi}$ towards the left of this line (see Fig. 6.2), and that of the straight part lies at $\frac{a}{2}$ distance towards right. Choosing O as origin and the *x*-axis towards right, Eq. 6.5 takes the form

$$\bar{x} = \frac{L_1\,\bar{x}_1 + L_2\,\bar{x}_2}{L}$$

$$0 = \frac{(\pi r)\left(-\frac{2r}{\pi}\right) + (a)\left(\frac{a}{2}\right)}{\pi r + a}$$

$$0 = -2r^2 + \frac{a^2}{2}$$

$$\therefore\; a = 2r$$

Example 6.17

A uniform rod is bent into the shape as shown in Fig. 6.29. Determine the coordinates of its centroid.

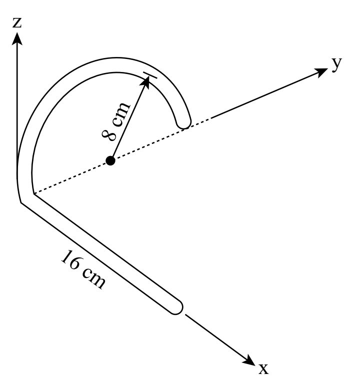

Figure 6.29 Figure for Ex. 6.17

Solution

The length of the straight part and the coordinates of its centroid are 16 cm and (8, 0, 0) cm, respectively. These are 8π cm and $\left(0, 8, \frac{16}{\pi}\right)$ cm for the circular part. For convenience, this problem would be solved in the tabular form given below.

Part	L_i	$\bar{x}_i$	$\bar{y}_i$	$\bar{z}_i$	$L_i\bar{x}_i$	$L_i\bar{y}_i$	$L_i\bar{z}_i$
Straight	16	8	0	0	128	0	0
Circular	8π	0	8	$\frac{16}{\pi}$	0	64π	128
Total	41.13				128	201.06	128

Equation 6.5 can now be used for finding out the coordinates of the centroid:

$$\bar{x} = \frac{\Sigma L_i\bar{x}_i}{L} = \frac{128}{41.13} = 3.11 \text{ cm}$$

$$\bar{y} = \frac{\Sigma L_i\bar{y}_i}{L} = \frac{201.06}{41.13} = 4.89 \text{ cm}$$

$$\bar{z} = \frac{\Sigma L_i\bar{z}_i}{L} = \frac{128}{41.13} = 3.11 \text{ cm}$$

Example 6.18

The homogeneous wire ABCD is bent as shown in Fig. 6.30. It is attached to a hinge at C. Determine the length l for which portion BCD of the wire remains horizontal. All dimensions are in mm.

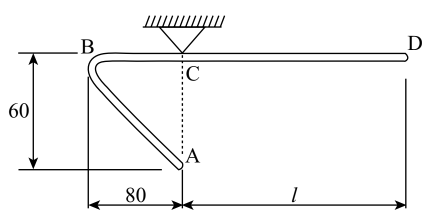

Figure 6.30 Figure for Ex. 6.18

Solution

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{60^2 + 80^2} = 100 \text{ mm}$$

For equilibrium to be possible in the position shown, the centroid of the bent wire must lie on line AC. Centroids of both AB and BC lie $\frac{80}{2}$ (= 40 mm) towards left of AC, and that of CD is at $\frac{l}{2}$ towards right. We choose C as the origin and CD as the x-axis.

Part	L_i	$\bar{x}_i$	$L_i\bar{x}_i$
AB	100	–40	–4000
BC	80	–40	–3200
CD	l	$\frac{l}{2}$	$\frac{l^2}{2}$
Total	$180 + l$		$\frac{l^2}{2} - 7200$

$$\bar{x} = \frac{\sum L_i \bar{x}_i}{L} = \frac{\frac{l^2}{2} - 7200}{180 + l}$$

For $\bar{x}$ to be zero, $\frac{l^2}{2} = 7200$

$$\therefore l = 120 \text{ mm}$$

Example 6.19

A wire is bent into a closed loop A–B–C–D–E–A as shown in Fig. 6.31. Portion AB is a circular arc of radius 5 m. Determine the centroid of the wire.

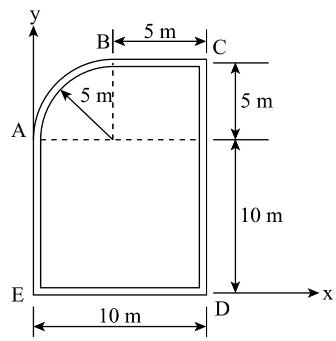

Figure 6.31 Figure for Ex. 6.19

Solution

We will use the result that the centroid of a quarter or a semicircular arc lies at a distance of $\frac{2r}{\pi}$ from its base (see Fig. 6.2).

Part	L_i	$\bar{x}_i$	$\bar{y}_i$	$L_i\ \bar{x}_i$	$L_i\ \bar{y}_i$
AB	$\frac{5\pi}{2}$	$5-\frac{10}{\pi}$	$10+\frac{10}{\pi}$	14.270	103.540
BC	5	7.5	15	37.5	75
CD	15	10	7.5	150	112.5
DE	10	5	0	50	0
EA	10	0	5	0	50
Total	47.854			251.77	341.04

$$\bar{x} = \frac{\sum L_i\ \bar{x}_i}{L} = \frac{251.77}{47.854} = 5.26 \text{ m}$$

$$\bar{y} = \frac{\sum L_i\ \bar{y}_i}{L} = \frac{341.04}{47.854} = 7.13 \text{ m}$$

Example 6.20

Determine the length of wire such that the centroid is located at O. Find the length in terms of *r* (Fig. 6.32).

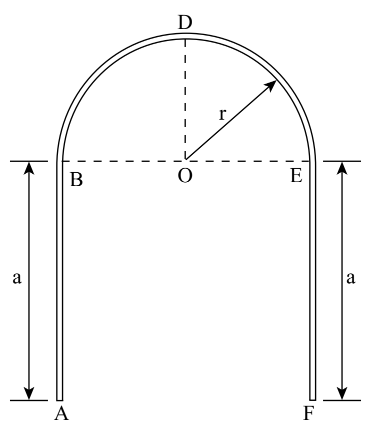

Figure 6.32 Figure for Ex. 6.20

Solution

Let the origin of the coordinate system be at O, with the *y*-axis pointing upwards.

$$\bar{y} = \frac{L_{BDE}\ \bar{y}_{BDE} + L_{AB}\ \bar{y}_{AB} + L_{EF}\ \bar{y}_{EF}}{L_{BDE} + L_{AB} + L_{EF}} = 0 \ \text{(Given)}$$

$$(\pi r)\left(\frac{2r}{\pi}\right)+a\left(-\frac{a}{2}\right)+a\left(-\frac{a}{2}\right)=0$$

$$2r^2 - a^2 = 0; \quad a = \sqrt{2}\, r$$

Example 6.21

Locate the centroid of the composite area shown in Fig. 6.33.

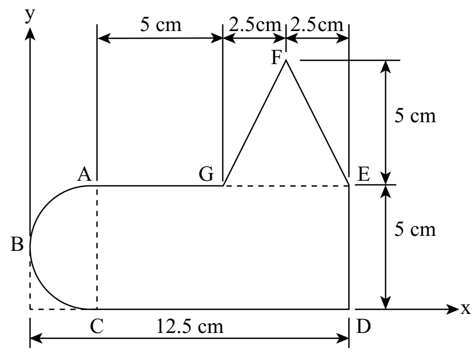

Figure 6.33 Figure for Ex. 6.21

Solution

We will use the results that the centroid of a semicircular disc of radius r lies at a distance of $\frac{4r}{3\pi}$ from its base (see Fig. 6.12), and that of a triangle of altitude h lies $\frac{h}{3}$ above its base (see Fig. 6.7).

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\ \bar{x}_i$	$A_i\ \bar{y}_i$
Semicircular sector ABC	$\frac{\pi \times 2.5^2}{2}$	$2.5 - \frac{4\times 2.5}{3\pi}$	2.5	14.127	24.544
Rectangle ACDE	50	7.5	2.5	375	125
Triangle EFG	12.5	10	$5+\frac{5}{3}$	125	83.333
Total	72.317			514.127	232.877

$$\bar{x} = \frac{\sum A_i\ \bar{x}_i}{A} = \frac{514.127}{72.317} = 7.11 \text{ cm}$$

$$\bar{y} = \frac{\sum A_i\ \bar{y}_i}{A} = \frac{232.877}{72.317} = 3.22 \text{ cm}$$

Example 6.22

A triangle is removed from a semicircular disc as shown in Fig. 6.34. Locate the centroid of the remaining part (shaded).

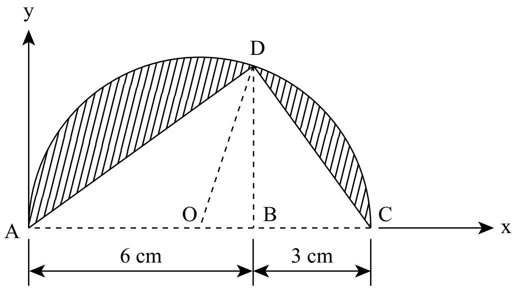

Figure 6.34 Figure for Ex. 6.22

Solution

Radius of the circle $= \frac{9}{2} = 4.5$ cm

Altitude (BD) of the triangle $= \sqrt{OD^2 - OB^2} = \sqrt{4.5^2 - (6 - 4.5)^2} = 4.243$ cm

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\ \bar{x}_i$	$A_i\ \bar{y}_i$
Semicircular disc	$\frac{\pi \times 4.5^2}{2}$	4.5	$\frac{4 \times 4.5}{3\pi}$	143.139	60.75
Triangle ABD	$-\frac{6 \times 4.243}{2}$	$6 - \frac{6}{3}$	$\frac{4.243}{3}$	–50.916	–18.003
Triangle BCD	$-\frac{3 \times 4.243}{2}$	$6 + \frac{3}{3}$	$\frac{4.243}{3}$	–44.552	–9.002
Total	12.715			47.671	33.745

$$\bar{x} = \frac{\sum A_i\ \bar{x}_i}{A} = \frac{47.671}{12.715} = 3.75 \text{ cm}$$

$$\bar{y} = \frac{\sum A_i\ \bar{y}_i}{A} = \frac{33.745}{12.715} = 2.65 \text{ cm}$$

Note that the concept of negative area has been used in this example because the shaded area is obtained by subtracting the areas of triangles ABD and BCD from the area of the semicircular disc. Subtracting an area is equivalent to adding a negative area of the same magnitude.

Example 6.23

Locate the centroid of the channel section shown in Fig. 6.35.

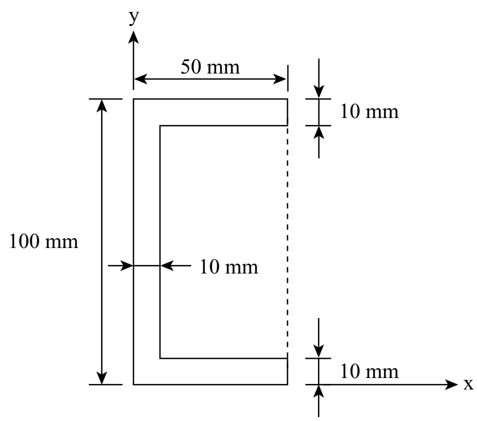

Figure 6.35 Figure for Ex. 6.23

Solution

This problem can be solved by considering three rectangles of areas 100 × 10, 40 × 10 and again 40 × 10 (other combinations are also possible). The other way is to consider the outer rectangle of area 100 × 50, and the inner rectangle of negative area 80 × 40. We will adopt the second approach since it would involve fewer calculations.

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\ \bar{x}_i$	$A_i\ \bar{y}_i$
Outer rectangle	5000	25	50	125000	250000
Inner rectangle	–3200	50–20	50	–96000	–160000
Total	1800			29000	90000

$$\bar{x} = \frac{\sum A_i\ \bar{x}_i}{A} = \frac{29000}{1800} = 16.11 \text{ mm}$$

$$\bar{y} = \frac{\sum A_i\ \bar{y}_i}{A} = \frac{90000}{1800} = 50 \text{ mm}$$

One may use the first approach also which would give the same answer. Take it as an exercise.

Note that we can use the symmetry argument to conclude that $\bar{y}$ is 50 mm.

Example 6.24

Determine the centroid of the shaded area shown in Fig. 6.36. All dimensions are in mm.

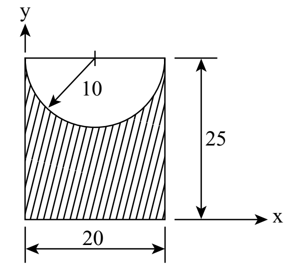

Figure 6.36 Figure for Ex. 6.24

Solution

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\ \bar{x}_i$	$A_i\ \bar{y}_i$
Rectangle	20×25	10	12.5	5000	6250
Semicircular sector	$-\dfrac{\pi \times 10^2}{2}$	10	$25 - \dfrac{4 \times 10}{3\pi}$	–1570.8	–3260.3
Total	342.9			3429.2	2989.7

$$\bar{x} = \frac{\sum A_i\ \bar{x}_i}{A} = \frac{3429.2}{342.9} = 10 \text{ mm}$$

$$\bar{y} = \frac{\sum A_i\ \bar{y}_i}{A} = \frac{2989.7}{342.9} = 8.7 \text{ mm}$$

Note that $\bar{x} = 10$ could have been concluded merely by symmetry argument without any calculations. It serves as a check that our calculations are correct.

Example 6.25

A semicircular area is removed from a trapezoid as shown in Fig. 6.37. Determine the centroid of the remaining area.

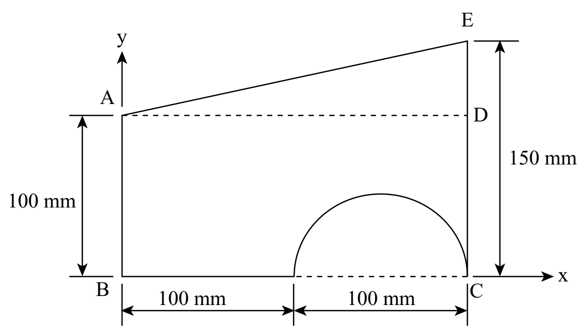

Figure 6.37 Figure for Ex. 6.25

Solution

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\ \bar{x}_i$	$A_i\ \bar{y}_i$
Rectangle ABCD	100×200	100	50	2000000	1000000
Triangle ADE	$\dfrac{200\times 50}{2}$	$\dfrac{2}{3}\times 200$	$100+\dfrac{50}{3}$	666666.67	583333.33
Semicircular sector	$-\dfrac{\pi\times 50^2}{2}$	150	$\dfrac{4\times 50}{3\pi}$	–589048.62	–83333.33
Total	21073			2077618	1500000

$$\bar{x} = \frac{\sum A_i\ \bar{x}_i}{A} = \frac{2077618}{21073} = 98.59 \text{ mm}$$

$$\bar{y} = \frac{\sum A_i\ \bar{y}_i}{A} = \frac{1500000}{21073} = 71.18 \text{ mm}$$

Example 6.26

Determine the centroid of the bracket of small and uniform thickness as shown in Fig. 6.38. All dimensions are in centimetre.

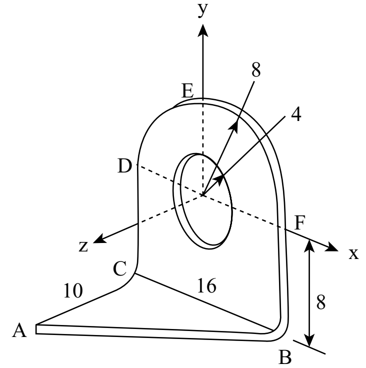

Figure 6.38 Figure for Ex. 6.26

Solution

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$\bar{z}_i$	$A_i\ \bar{x}_i$	$A_i\ \bar{y}_i$	$A_i\ \bar{z}_i$
Rectangle BCDF	8×16	0	–4	0	0	–512	0
Triangle ABC	$\frac{10\times16}{2}$	$-\left(8-\frac{16}{3}\right)$	–8	$\frac{10}{3}$	–213.33	–640	266.67
Semicircle DEF	$\frac{\pi\times8^2}{2}$	0	$\frac{4\times8}{3\pi}$	0	0	341.33	0
Circular hole	$-\pi\times4^2$	0	0	0	0	0	0
Total	258.27				–213.33	–810.67	266.67

$$\bar{x} = \frac{\sum A_i\ \bar{x}_i}{A} = -\frac{213.33}{258.27} = -0.83 \text{ cm}$$

$$\bar{y} = \frac{\sum A_i\ \bar{y}_i}{A} = -\frac{810.67}{258.27} = -3.14 \text{ cm}$$

$$\bar{z} = \frac{\sum A_i\ \bar{z}_i}{A} = \frac{266.67}{258.27} = 1.03 \text{ cm}$$

Example 6.27

Locate the centroid of the frustrum of the cone shown in Fig. 6.24 (figure for Ex. 6.14) using the concept of composite volumes.

Solution

We will use the result that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$, and its centroid lies $\frac{h}{4}$ distance above the base.

From similar triangles,

$$\frac{b'}{b} = \frac{h-\frac{h}{2}}{h} \quad \therefore b' = \frac{b}{2}$$

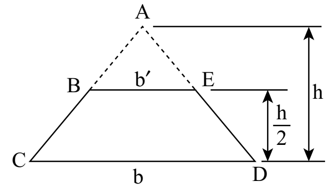

Figure 6.39 Centroid calculation using the concept of composite volumes in Ex. 6.27

Part	V_i	$\bar{y}_i$	$V_i\ \bar{y}_i$
Cone ACD	$\frac{1}{3}\pi\left(\frac{b}{2}\right)^2 h$	$\frac{h}{4}$	$\frac{\pi b^2 h^2}{48}$
Cone ABE	$-\frac{1}{3}\pi\left(\frac{b}{4}\right)^2 \frac{h}{2}$	$\frac{h}{2}+\frac{h}{8}$	$-\frac{\pi b^2 h^2}{96}\cdot\frac{5}{8}$
Total	$\frac{1}{3}\pi b^2 h\left(\frac{1}{4}-\frac{1}{32}\right)$		$\frac{\pi b^2 h^2}{48}\left(1-\frac{5}{16}\right)$

$$\bar{y}=\frac{\sum V_i\ \bar{y}_i}{V}=\frac{\frac{\pi b^2 h^2}{48}\cdot\frac{11}{16}}{\frac{1}{3}\pi b^2 h\cdot\frac{7}{32}}=\frac{11\times 3\times 32}{48\times 16\times 7}h=\frac{11h}{56}$$

Example 6.28

A hole of diameter d_1 is drilled axially up to depth h ($h < L$) in a cylinder of diameter d_2 and length L, as shown in Fig. 6.40. Determine h for which the centroid of the drilled cylinder lies nearest to its base.

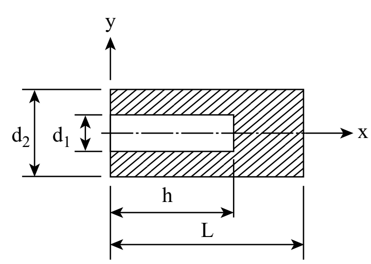

Figure 6.40 Figure for Ex. 6.28

Solution

The centroid of the solid (i.e., undrilled) cylinder lies at $\bar{x}=\frac{L}{2}$. As it is drilled, the centroid starts shifting to the right. When drilling is full ($h = L$), the centroid is again at $\bar{x}=\frac{L}{2}$. Therefore, at a certain value of h, the centroid would lie maximum to the right (i.e., nearest to the bottom of the cylinder). This can be found out by differentiating $\bar{x}$ with respect to h, and equating it to zero.

Part	V_i	$\bar{x}_i$	$V_i\,\bar{x}_i$
Full cylinder	$\frac{\pi}{4}d_2^2 L$	$\frac{L}{2}$	$\frac{\pi\, d_2^2 L^2}{8}$
Hole	$-\frac{\pi}{4}d_1^2 h$	$\frac{h}{2}$	$-\frac{\pi d_1^2 h^2}{8}$
Total	$\frac{\pi}{4}\left(d_2^2 L - d_1^2 h\right)$		$\frac{\pi}{8}\left(d_2^2 L^2 - d_1^2 h^2\right)$

$$\bar{x} = \frac{\sum V_i\,\bar{x}_i}{V} = \frac{\frac{\pi}{8}\left(d^2 L^2 - d_1^2 h^2\right)}{\frac{\pi}{4}\left(d_2^2 L - d_1^2 h\right)} = \frac{d_2^2 L^2 - d_1^2 h^2}{2\left(d_2^2 L - d_1^2 h\right)}$$

$$\frac{d\bar{x}}{dh} = \frac{-2d_1^2 h\cdot 2\left(d_2^2 L - d_1^2 h\right) - \left(d_2^2 L^2 - d_1^2 h^2\right)\left(-2d_1^2\right)}{4\left(d_2^2 L - d_1^2 h\right)^2}$$

$$\frac{d\bar{x}}{dh} = 0 \Rightarrow -2h\left(d_2^2 L - d_1^2 h\right) + \left(d_2^2 L^2 - d_1^2 h^2\right) = 0$$

$$d_1^2\, h^2 - 2\, d_2^2 L h + d_2^2\, L^2 = 0$$

$$h = \frac{2d_2^2\; L \pm \sqrt{4d_2^4\;\, L^2 - 4d_1^2\; d_2^2\; L^2}}{2d_1^2} = \left(\frac{d_2}{d_1}\right)^2 L\left[1 \pm \sqrt{1 - \left(\frac{d_1}{d_2}\right)^2}\right]$$

It can be argued in a number of ways that the ± symbol can only be a negative sign in the obtained expression for h. Given below is one such argument.

Since h is less than L,

$$\left(\frac{d_2}{d_1}\right)^2\left[1 \pm \sqrt{1 - \left(\frac{d_1}{d_2}\right)^2}\right] < 1$$

Further, since $d_1 < d_2$, $\frac{d_2}{d_1}$ and hence $\left(\frac{d_2}{d_1}\right)^2$ is greater than 1. Therefore,

$$1 \pm \sqrt{1 - \left(\frac{d_1}{d_2}\right)^2} < 1$$

$$\pm\sqrt{1-\left(\frac{d_1}{d_2}\right)^2} < 0$$

which is possible only with a negative sign. Therefore,

$$h = \left(\frac{d_2}{d_1}\right)^2 L\left[1-\sqrt{1-\left(\frac{d_1}{d_2}\right)^2}\right]$$

6.6 PAPPUS THEOREM

Pappus Theorem is also referred to as Pappus-Guldinus Theorems. The first of these theorems relates a surface of revolution to its generating curve, whereas the second theorem relates a volume of revolution to its generating area.

The first theorem states that *the surface of revolution developed by revolving a coplanar generating curve about an axis of revolution has an area equal to the length of the generating curve times the circumference of the circle formed by the centroid of the generating curve in the process of generating the surface of revolution*. The generating curve may touch the axis of revolution, but must not cross it.

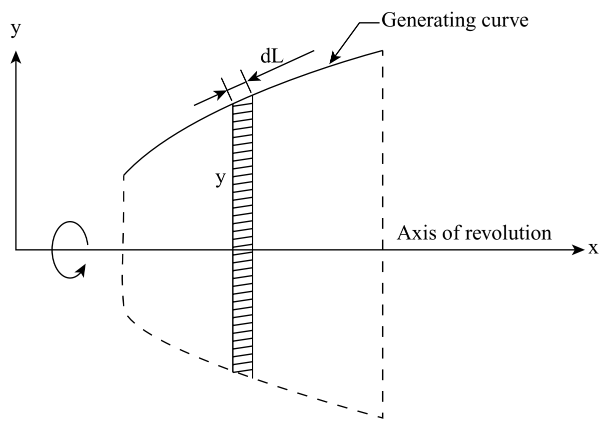

Figure 6.41 A coplanar generating curve being revolved about x-axis

To prove this theorem, consider the generating curve shown in Fig. 6.41 which is being revolved about the x-axis. The strip, shown as the shaded area in the figure, formed by the elemental length dL of the generating curve in one revolution, has an area given by

$$dA = (2\pi y)\, dL$$

The area of the surface of revolution would be

$$A = \int dA = 2\pi \int y\, dL = 2\pi \bar{y} L \quad \text{(From Eq. 6.2)} \tag{6.8}$$

which proves the theorem (L is the length of the curve, and $\bar{y}$ is the y-coordinate of the centroid of the curve).

If the generating curve is composed of several simple curves with known centroids, such as the one shown in Fig. 6.42, then Eq. 6.8 takes the form

$$A = 2\pi \sum \bar{y}_i L_i \tag{6.9}$$

where L_i is the length of a segment, and $\bar{y}_i$ is the y-coordinate of its centroid.

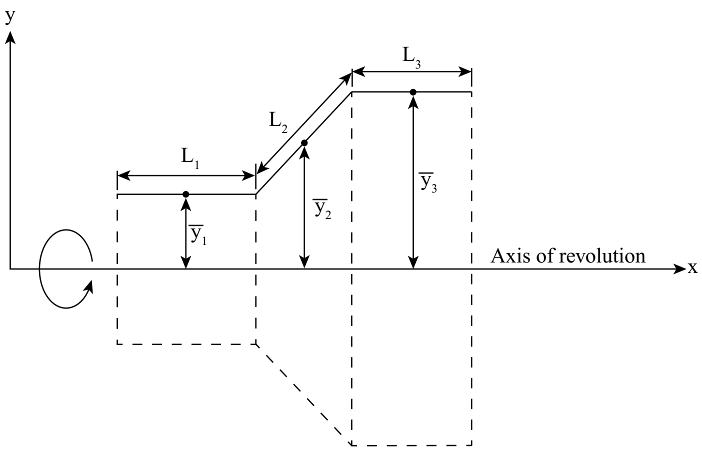

Figure 6.42 Surface of revolution generated by a composite generating curve

The second theorem states that *the solid of revolution developed by revolving a plane surface about a coplanar axis of revolution has a volume equal to the area of the surface times the circumference of the circle formed by the centroid of the surface in the process of generating the solid of revolution*. The generating surface may touch the axis of revolution, but must not cross it.

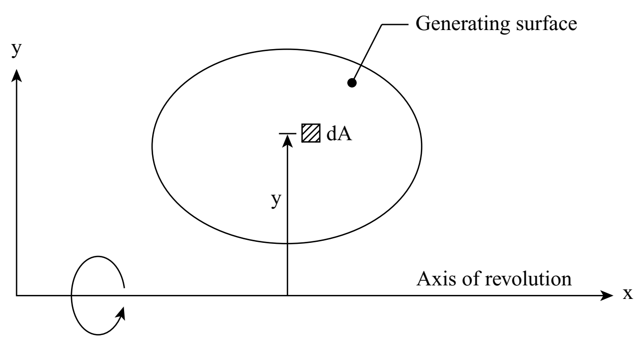

Figure 6.43 Solid of revolution generated by revolving a plane surface about *x*-axis

To prove this theorem, consider the generating surface shown in Fig. 6.43 which is being revolved about the *x*-axis. The volume of the solid generated by the elemental area *dA* in one revolution,

$$dV = (2\pi y)\, dA$$

The total volume of the solid of revolution would be

$$V = \int dV = 2\pi \int y\, dA = 2\pi \bar{y} A \quad \text{(From Eq. 6.3)} \tag{6.10}$$

which proves the theorem (A is the area of the surface, and $\bar{y}$ is the y-coordinate of the centroid of the surface).

If the generating surface is made up of several simple areas with known centroids, such as the one shown in Fig. 6.44, then Eq. 6.10 takes the form

$$V = 2\pi \sum \bar{y}_i A_i \tag{6.11}$$

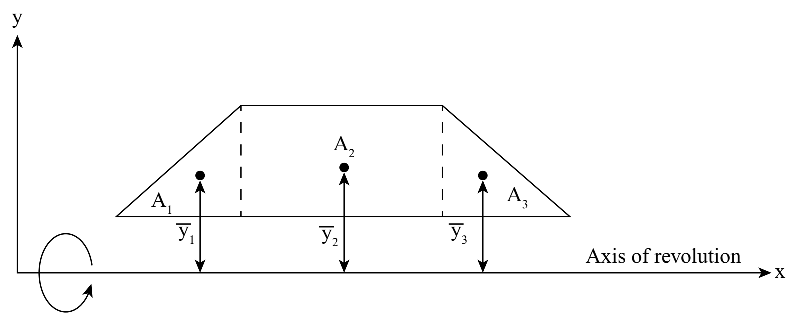

Figure 6.44 Solid of revolution generated by a composite generating surface

Example 6.29

Calculate the lateral area of a cone of base radius *r* and slant length *l*, using Pappus theorem.

Solution

The surface of the cone can be generated by revolving the slant line about the *y*-axis (Fig. 6.45).

From similar triangles,

$$\frac{\bar{x}}{r} = \frac{l/2}{l}; \quad \bar{x} = \frac{r}{2}$$

From Pappus theorem,

$$A = 2\pi \bar{x}\, l = \pi r l$$

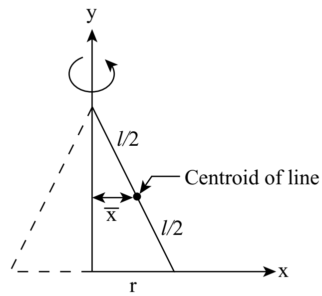

Figure 6.45 Surface of a cone generated by revolving a slant line about *y*-axis

Example 6.30

Compute the volume of a cone of base radius *r* and height *h*, using pappus theorem.

Solution

The solid cone can be generated by revolving the triangle about the *y*-axis (Fig. 6.46).

From Pappus theorem,

$$V = 2\pi \bar{x} A = 2\pi\left(\frac{r}{3}\right)\left(\frac{1}{2}rh\right) = \frac{1}{3}\pi r^2 h$$

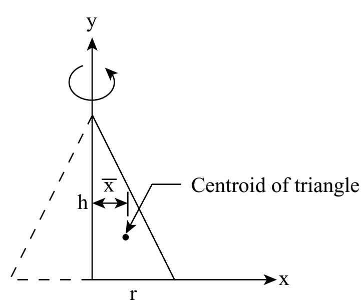

Figure 6.46 Solid cone generated by revolving a triangle about *y*-axis

Example 6.31

Determine the volume of the solid shown in Fig. 6.26 by Pappus theorem.

Solution

Since the angle of revolution is 180°, Eq. 6.11 takes the from

$$V = \pi \sum \bar{y}_i A_i$$

The generating surface is composed of a rectangular and a triangular areas with known centroids (Fig. 6.47). Therefore,

$$V = \pi\left[\frac{a}{4}\times\left(h\times\frac{a}{2}\right)+\left(\frac{a}{2}+\frac{1}{3}\times\frac{a}{2}\right)\times\left(\frac{1}{2}\times h\times\frac{a}{2}\right)\right]$$

$$= \pi ha^2\left(\frac{1}{8}+\frac{2}{3}\times\frac{1}{4}\right)=\frac{7}{24}\pi a^2 h$$

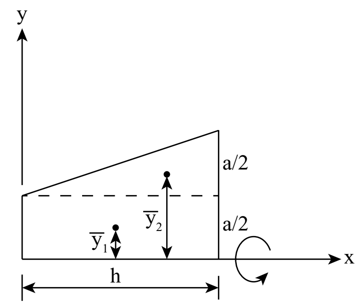

Figure 6.47 Composite area consisting of a rectangular and a triangular part

Example 6.32

Determine the volume of the solid in Prob. 6.12 by Pappus theorem.

Solution

The generating surface is a composite area consisting of a triangular part and a circular segment (Fig. 6.48).

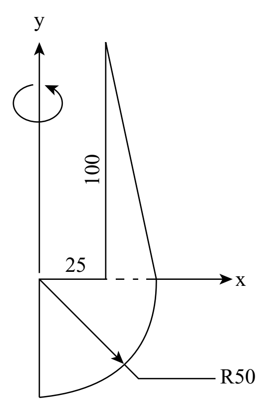

Figure 6.48 Composite area consisting of a triangular and a circular area

$$A_{\text{triangle}} = \frac{1}{2}\times 100\times 25\,\text{mm}^2$$

$$\overline{}_{\text{triangle}} = 25+\frac{25}{}\,\text{mm}$$

$$A_{\text{circular segment}} = \frac{\pi\times 50^2}{4}\,\text{mm}^2$$

$$\bar{x}_{\text{circular segment}} = \frac{4r}{3\pi} = \frac{4\times 50}{3\pi}\,\text{mm}$$

$$V = 2\pi\left(\bar{x}_{\text{triangle}}\times A_{\text{triangle}} + \bar{x}_{\text{circular segment}}\times A_{\text{circular segment}}\right)$$

$$= 2\pi\left[\left(25+\frac{25}{3}\right)\times\frac{1}{2}\times 100\times 25+\frac{4\times 50}{3\pi}\times\frac{\pi\times 50^2}{4}\right]$$

$$= 523598.8\ \text{mm}^3$$

Problems

6.1 Locate the centroid of the area shown in the figure.

Ans. $\bar{x} = \dfrac{3a}{4};\ \bar{y} = \dfrac{3b}{10}$

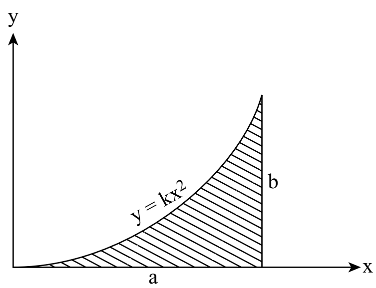

Figure Prob. 6.1

6.2 Locate the centroid of the area bounded by the line $y = ax$ and the parabola $y = x^2$. a is a positive constant.

Ans. $\bar{x} = \dfrac{a}{2};\ \bar{y} = \dfrac{2a^2}{5}$

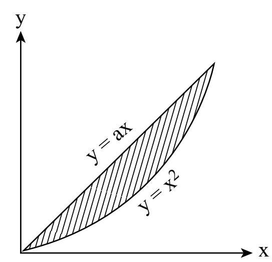

Figure Prob. 6.2

6.3 Locate the centroid of the area bounded by the parabola $y = a - \dfrac{x^2}{4a}$ and the x-axis.

Ans. $\bar{x} = 0;\ \bar{y} = \dfrac{2a}{5}$

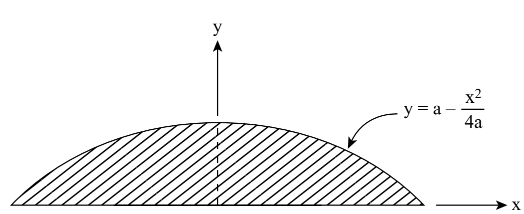

Figure Prob. 6.3

6.4 The shaded area bounded by the parabola $y^2 = ax$, line $x = a$, and the x-axis is revolved fully about the x-axis to form a paraboloid of revolution. Locate its x-centroid.

Ans. $\bar{x} = \dfrac{2a}{3}$

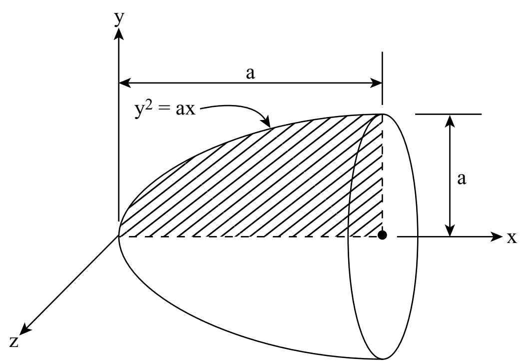

Figure Prob. 6.4

6.5 Locate the centroid of a quarter-cone of base radius r and height h.

Ans. $\bar{x} = \bar{y} = \dfrac{r}{\pi}$; $\bar{z} = \dfrac{h}{4}$

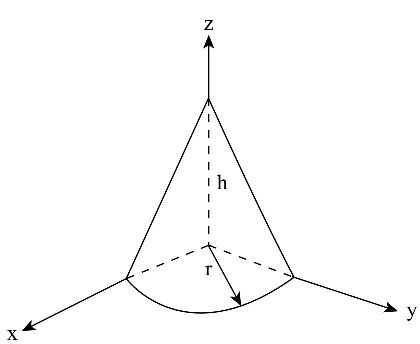

Figure Prob. 6.5

6.6 Locate the x-coordinate of the centroid of the shaded area which is bounded by the two circles of the same radii a, and the y-axis.

Ans. $\bar{x} = 0.3392\ a$

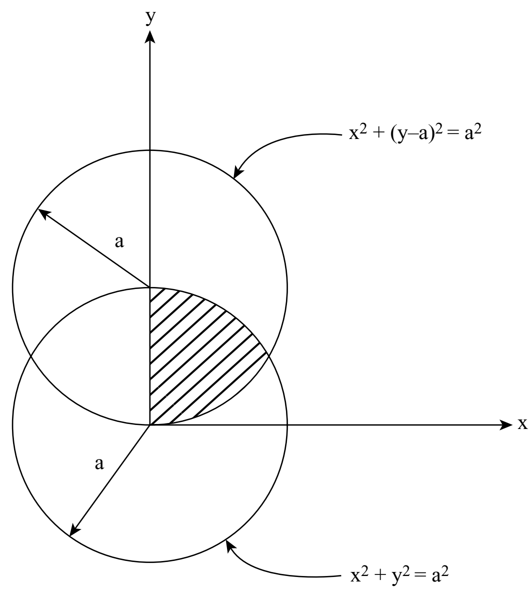

Figure Prob. 6.6

6.7 A quarter circle of radius a, with the centre at (a, 0) in the xy-plane, (the shaded area in the figure) is revolved fully about the y-axis. Determine the y-coordinate of the centroid of the solid thus formed.

Ans. $\bar{y} = 0.41\ a$

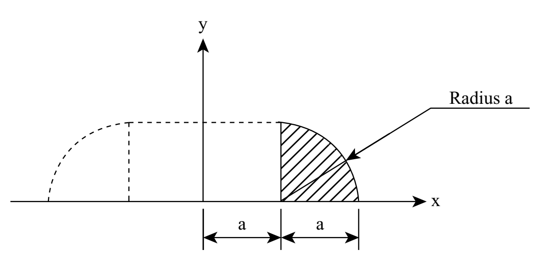

Figure Prob. 6.7

6.8 A thin wire is bent in the shape shown. Determine the *y*-coordinate of its centroid. All dimensions are in centimetre.

Ans. $\bar{y} = 13.57$ cm

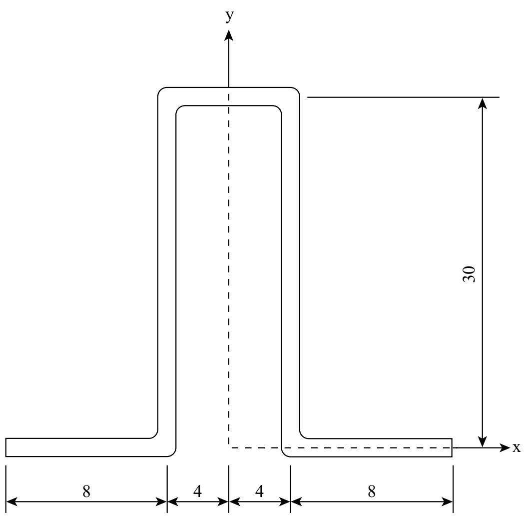

Figure Prob. 6.8

6.9 A wire is bent in the shape shown. Locate its centroid. The semicircular arc lies in the *xy*-plane with the centre at (30, 0, 0).

Ans. $\bar{x} = 22.76$ mm; $\bar{y} = -11.27$ mm
$\bar{z} = -0.40$ mm

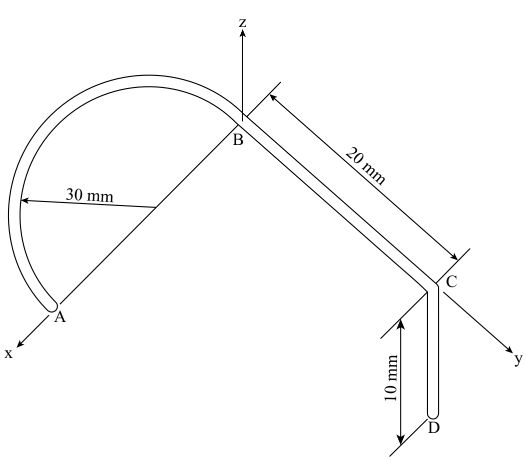

Figure Prob. 6.9

6.10 Locate the centroid in Ex. 6.9 using the concept of composite areas.

Ans. $\bar{x} = \bar{y} = 0.2234\,a$

6.11 A circular disk of diameter *D* has three holes of diameter *d* in the position shown. A fourth hole is to be drilled at the same radial distance *r* (the pitch-circle radius) so that the centroid of the disk lies at its centre O. Determine the required diameter and the angular position of the fourth hole.

Ans. Diameter = 1.227 *d*; $\theta = 84.9°$

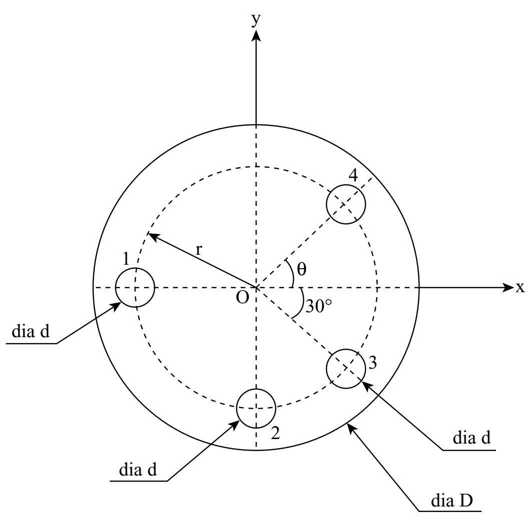

Figure Prob. 6.11

6.12 The area shown in the figure is revolved fully about the *y*-axis to form a solid of revolution. Determine the *y*-coordinate of the centroid. All dimensions are in millimeter.

Ans. $\bar{y} = 6.25$ mm

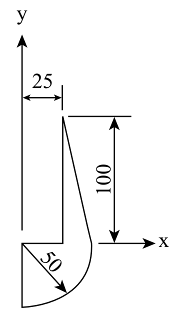

Figure Prob. 6.12

6.13 A top is made up of a cone of base diameter 24 mm, height 60 mm and a hemisphere of diameter 24 mm. Determine the *y*-coordinate of the centroid of the top.

Ans. $\bar{y} = 50.57$ mm

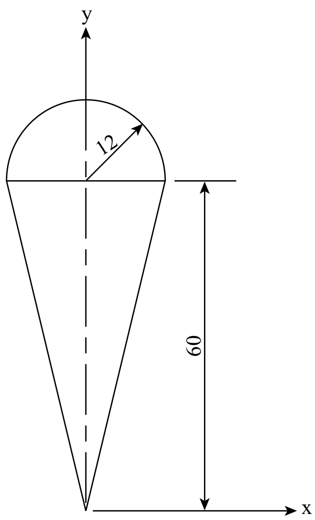

Figure Prob. 6.13