Laplace Transform

CHAPTER 4

LEARNING OUTCOMES

At the end of this chapter students should be able to:

- Use the Definition of Laplace Transform
- Use Table of Laplace Transforms
- Understand Inverse Laplace Transform
- Define Laplace Transform of Derivatives

4.0 Introduction

In this analysis, Laplace Transform is often explained as a conversion from the time-domain (t) to the frequency-domain (s). Laplace Transform is a generally used integral transform for solving differential and integral equations. Laplace transform is widely used in engineering applications. The function of this transform provides a systematic alternative method for solving differential equations.

4.1 Using the Definition of Laplace Transform

The Laplace Transform of a function $f(t)$ for all real numbers $t \geq 0$, function $F(s)$, is defined as:

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

The commonly used notations for the Laplace Transform are as follows:

- $L\{f(t)\}$ or $\mathcal{L}\{f(t)\}$
- $L(f)$ or $\mathcal{L}f$
- $\overline{f}(s)$ or $f(s)$

4.1.1 Deriving the Laplace Transform of an Expression by Using the Integral Definition

Given that $f(t)=a$ where a is a constant (can be any number and alphabet except s and t).

$$F(s) = \int_0^\infty e^{-st}(a)\ dt$$
$$= a\int_0^\infty e^{-st}\ dt$$
$$= a\left[\frac{e^{-st}}{-s}\right]_0^\infty$$
$$= a\left[\frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s}\right]$$
$$= a\left[0+\frac{1}{s}\right]$$
$$= \frac{a}{s}$$

Given that $f(t)=e^{at}$ where a is a constant (can be any number and alphabet except s and t).

$$F(s) = \int_0^\infty e^{-st}\cdot e^{at}\,dt$$
$$= \int_0^\infty e^{-(s-a)t}\,dt$$
$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]$$
$$= \frac{e^{-(s-a)\infty}}{-(s-a)} - \frac{e^{-(s-a)0}}{-(s-a)}$$
$$= 0+\frac{1}{s-a}$$
$$= \frac{1}{s-a}$$

$$F(s) = \int_0^\infty e^{-st}\cdot e^{-at}\,dt$$
$$= \int_0^\infty e^{-(s+a)t}\,dt$$
$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]$$
$$= \frac{e^{-(s+a)\infty}}{-(s+a)} - \frac{e^{-(s+a)0}}{-(s+a)}$$
$$= 0+\frac{1}{s+a}$$
$$= \frac{1}{s+a}$$

Given that $f(t) = t$.

$$F(s) = \int_0^\infty e^{-st} \cdot t \; dt$$
$$= \int_0^\infty te^{-st} dt....(1)$$

Using integration by parts; $\int u\,dv = uv - \int v\,du$

Priority to choose u:

1. Logarithm
2. Algebra
3. Exponent

$$u = t; \qquad \int dv = \int e^{-st}\; dt$$
$$\frac{du}{dt} = 1; \qquad v = -\frac{1}{s}e^{-st}$$
$$du = dt$$

$$\therefore F(s) = \left[-\frac{t}{s}e^{-st}\right]_0^\infty + \int_0^\infty -\frac{1}{s}e^{-st} dt$$
$$= \left[-\frac{t}{s}e^{-st}\right]_0^\infty - \frac{1}{s}\int_0^\infty e^{-st} dt$$
$$= \left[-\frac{t}{se^{st}} - \frac{1}{s^2}e^{-st}\right]_0^\infty$$
$$= \left[-\frac{t}{se^{st}} - \frac{e^{-st}}{s^2}\right]_0^\infty$$
$$= (0) - \left(-0 - \frac{1}{s^2}\right)$$
$$= \frac{1}{s^2}$$

Given that $f(t) = \sin at$ where a is a constant (any number and alphabet except s and t).

$$F(s) = \int_o^\infty e^{-st} \; \sin at \; dt \; ....(1)$$

Using integration by parts; $\int u\,dv = uv - \int v\,du$

Priority to choose u:

1. Logarithm
2. Algebra
3. Exponent

$$u = e^{-st} \quad ; \quad \int dv = \int \sin at \; dt$$

$$\frac{du}{dt} = -se^{-st} \qquad v = -\frac{1}{a}\cos at$$

$$du = -se^{-st}\,dt$$

$$\therefore F(s) = -\frac{e^{-st}}{a}\cos at - \int -\frac{1}{a}\cos at\left(-se^{-st}\right)dt$$

$$= -\frac{e^{-st}}{a}\cos at - \frac{s}{a}\int e^{-st}\cos at \; dt \qquad \text{........(2)}$$

From (2) assume:

$$A = \int e^{-st}\cos at \; dt$$

$$u = e^{-st} \quad ; \quad \int dv = \int \cos at$$

$$\frac{du}{dt} = -se^{-st} \qquad v = \frac{1}{a}\sin at$$

$$du = -se^{-st}\,dt$$

$$A = \frac{e^{-st}}{a}\sin at - \int \frac{1}{a}\sin at\left(-se^{-st}\right)dt$$

$$= \frac{e^{-st}}{a}\sin at + \frac{s}{a}\int e^{-st}\sin at \; dt \qquad \text{........(3)}$$

Replace (1) in (3) so, $$A = \frac{e^{-st}}{a}\sin at + \frac{s}{a}F(s) \qquad \text{........(4)}$$

Replace (4) in (2) so, $$F(s) = -\frac{e^{-st}}{a}\cos at - \frac{s}{a}\left\{\frac{e^{-st}}{a}\sin at + \frac{s}{a}F(s)\right\}$$

$$F(s) = -\frac{e^{-st}}{a}\cos at - \frac{se^{-st}}{a^2}\sin at - \frac{s^2}{a^2}F(s)$$

$$F(s) + \frac{s^2}{a^2}F(s) = -\frac{e^{-st}}{a}\cos at - \frac{se^{-st}}{a^2}\sin at$$

$$\left\{\frac{a^2+s^2}{a^2}\right\} F(s) = -\frac{e^{-st}}{a}\left\{\cos at+\frac{s}{a}\sin at\right\}$$

$$F(s) = \left[-\frac{ae^{-st}(\cos at+\frac{s}{a}\sin at)}{s^2+a^2}\right]_0^\infty$$

$$= 0-\left[-\frac{ae^{0}(\cos 0+\frac{s}{a}\sin 0)}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2}$$

Given that $f(t)=\cos at$ where a is a constant (any number and alphabet except s and t)

$$F(s) = \int_0^\infty e^{-st} \cos at \; dt \qquad \text{........(1)}$$

Using integration by parts; $\int uv\,du = uv - \int v\,du$

$$u = e^{-st}; \qquad \int dv = \int \cos at \; dt$$

$$\frac{du}{dt} = -se^{-st} \qquad v = \frac{1}{a}\sin at$$

$$du = -se^{-st} \; dt$$

$$F(s) = \frac{e^{-st}}{a}\sin at \; - \int \frac{1}{a}\sin at(-se^{-st}) \; dt$$

$$F(s) = \frac{e^{-st}}{a} \sin at \; + \frac{s}{a}\int e^{-st} \sin at \; dt \qquad \text{........(2)}$$

From (2), assume,

$$A = \int e^{-st} \sin at \; dt$$

$$u = e^{-st} \qquad ; \qquad \int dv = \int \sin at \ dt$$

$$\frac{du}{dt} = -se^{-st} \qquad v = -\frac{1}{a}\cos at$$

$$du = -se^{-st} \ dt$$

$$A = -\frac{e^{-st}}{a}\cos at - \int -\frac{1}{a}\cos at\left(-se^{-st}\right)dt$$

$$= -\frac{e^{-st}}{a}\cos at - \frac{s}{a}\int e^{-st}\cos at \ \ dt \qquad ....(3)$$

Replace (1) in (3) so, $$A = -\frac{e^{-st}}{a}\cos at - \frac{s}{a}F(s) \qquad ....(4)$$

Replace (4) in (2) so, $$F(s) = \frac{e^{-st}}{a}\sin at + \frac{s}{a}\left(-\frac{e^{-st}}{a}\cos at - \frac{s}{a}F(s)\right)$$

$$F(s) = \frac{e^{-st}}{a}\sin at - \frac{se^{-st}}{a^2}\cos at - \frac{s^2}{a^2}F(s)$$

$$F(s) + \frac{s^2F(s)}{a^2} = \frac{e^{-st}}{a}\sin at - \frac{se^{-st}}{a^2}\cos at$$

$$\left(\frac{a^2+s^2}{a^2}\right)F(s) = \frac{e^{-st}}{a}\left(\sin at - \frac{s}{a}\cos at\right)$$

$$F(s) = \left[\frac{ae^{-st}}{s^2+a^2}\left(\sin at - \frac{s}{a}\cos at\right)\right]_0^\infty$$

$$F(s) = 0 - \left[\frac{ae^0}{s^2+a^2}\left(\sin 0 - \frac{s}{a}\cos 0\right)\right]$$

$$= -\frac{a}{s^2+a^2}\left(-\frac{s}{a}\right)$$

$$= \frac{s}{s^2+a^2}$$

Example 4.1

Find the Laplace Transform for $f(t)=4$ by using the Definition of Laplace Transform.

Solution

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st}\,(4)\;dt \\
&= 4\int_0^\infty e^{-st}\;dt \\
&= 4\left[\frac{e^{-st}}{-s}\right]_0^\infty \\
&= 4\left[\frac{e^{-s(\infty)}}{-s}-\frac{e^{-s(0)}}{-s}\right] \\
&= 4\left[0+\frac{1}{s}\right] \\
&= \frac{4}{s}
\end{aligned}
$$

Example 4.2

Find the Laplace Transform for $f(t)=e^{4t}$ by using the Definition of Laplace Transform.

Solution

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st}\cdot e^{4t}\;dt \\
&= \int_0^\infty e^{-(s-4)t}\;dt \\
&= \left[\frac{e^{-(s-4)t}}{-(s-4)}\right]_0^\infty \\
&= \frac{e^{-(s-4)\infty}}{-(s-4)}-\frac{e^{-(s-4)0}}{-(s-4)} \\
&= 0+\frac{1}{s-4} \\
&= \frac{1}{s-4}
\end{aligned}
$$

Quick Check 4.1

Find the Laplace Transform for the following function by using the Definition of Laplace Transform.

1 $f(t)=1$

2 $f(t)=-7$

3 $f(t)=e^{-3t}$

4 $f(t)=\frac{1}{3}e^{-2t}+\frac{1}{2}$

4.2 Using Table of Laplace Transforms

4.2.1 Obtaining the Laplace Transform by Using Table of Laplace Transforms

We have discussed extensively in the preceding section on how to obtain the Laplace transforms by using integration method. However, the approach using the definition of Laplace transforms is tedious and it would be burdensome to obtain the transform using the definition for each problem. Fortunately, by taking advantage of many general properties that are helpful, together with the lists in the table, we can obtain the transform of most functions easily. The following Table of Laplace Transforms is very useful when solving problems in science and engineering that require Laplace transform. Each expression in the right-hand column is derived by finding the infinite integral that we saw in the Definition of Laplace Transform section before.

Table 4.1 Table of Laplace Transforms

No	$f(t)$	$F(s)$	No	$f(t)$	$F(s)$
1.	a	$\frac{a}{s}$	5.	e^{-at}	$\frac{1}{s+a}$
2.	at	$\frac{a}{s^2}$	6.	te^{-at}	$\frac{1}{(s+a)^2}$
3.	t^n	$\frac{n!}{s^{n+1}}$	7.	$t^n.e^{at}, n=1,2,3\ldots$	$\frac{n!}{(s-a)^{n+1}}$
4.	$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	8.	$t^n.f(t)$	$(-1)^n\frac{d^n}{ds^n}\left[F(s)\right]$

No	$f(t)$	$F(s)$	No	$f(t)$	$F(s)$
9.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	16.	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
10.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	17.	$e^{at} \sinh \omega t$	$\frac{\omega}{(s-a)^2-\omega^2}$
11.	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	18.	$e^{-at} \sinh \omega t$	$\frac{\omega}{(s+a)^2-\omega^2}$
12.	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	19.	$e^{-at} \cosh \omega t$	$\frac{s+a}{(s+a)^2-\omega^2}$
13.	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	20.	$f_1(t)+f_2(t)$	$F_1(s)+F_2(s)$
14.	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	21.	$\int_0^t f(u)du$	$\frac{F(s)}{s}$
15.	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$	22.	$f(t-a)u(t-a)$	$e^{-as}F(s)$

Trigonometry Identity

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin 2x = 2 \sin x \cos x$
3. $\cos 2x = 2\cos^2 x - 1; \quad \cos 2x = 1 - 2\sin^2 x$

4.2.2 | Applying Laplace Transform Properties

Theorem 1: Linearity

If $L\{f_1(t)\}$ and $L\{f_2(t)\}$ exist, and if a and b are constants then $L\{af_1(t)+bf_2(t)\}$ $=aL\{f_1(t)\}+bL\{f_2(t)\}$

Proof: Using the definition of Laplace transform

$$L\{af_1(t)+bf_2(t)\} = \int_0^\infty e^{-st}\left[af_1(t)+bf_2(t)\right]dt$$

$$= \int_0^\infty ae^{-st} f_1(t)dt + \int_0^\infty be^{-st} f_2(t)dt$$

$$= a\int_0^\infty e^{-st} f_1(t)dt + b\int_0^\infty e^{-st} f_2(t)dt$$

$$= a\,L\{f_1(t)\} + b\,L\{f_2(t)\} \text{ where } a \text{ and } b \text{ are constants}$$

Example 4.3

Find the Laplace Transforms of the following functions by using Table of Laplace Transforms.

(a) $f(t)=3e^{2t}+4\sin 5t$ (b) $f(t)=\sinh at$ (c) $f(t)=\cosh at$ (d) $f(t)=\sin^2 \omega t$

(e) $f(t)=t^4+3t^2-5$ (f) $f(t)=\cosh 4t+\cos 3t$ (g) $f(t)=(3t-1)^2$

Solution

(a)
$$L\{3e^{2t}+4\sin 5t\} = L\{3e^{2t}\}+L\{4\sin 5t\}$$
$$= 3\,L\{e^{2t}\}+4\,L\{\sin 5t\}$$
$$= 3\left(\frac{1}{s-2}\right)+4\left(\frac{5}{s^2+25}\right)$$
$$= \frac{3}{s-2}+\frac{20}{s^2+25}$$

(b)
$$L\{\sinh at\} = \frac{a}{s^2-a^2}$$

(c)
$$L\{\cosh at\} = \left(\frac{s}{s^2-a^2}\right)$$

(d)
$$L\{\sin^2 \omega t\} = L\left\{\frac{1}{2}(1-\cos 2\omega t)\right\}$$
$$= L\left\{\frac{1}{2}\right\}-L\left\{\frac{1}{2}\cos 2\omega t\right\}$$
$$= L\left\{\frac{1}{2}\right\}-\frac{1}{2}\,L\{\cos 2\omega t\}$$
$$= \frac{1}{2s}-\frac{1}{2}\left(\frac{s}{s^2+4\omega^2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4\omega^2}\right)$$
$$= \frac{1}{2}\left(\frac{s^2+4\omega^2-s^2}{s(s^2+4\omega^2)}\right)$$
$$= \frac{2\omega^2}{s(s^2+4\omega^2)}$$

(e)
$$L\{t^4+3t^2-5\} = L\{t^4\}+L\{3t^2\}+L\{-5\}$$
$$= \left(\frac{4!}{s^{4+1}}\right)+3\left(\frac{2!}{s^{2+1}}\right)+\left(\frac{-5}{s}\right)$$
$$= \frac{24}{s^5}+\frac{6}{s^3}-\frac{5}{s}$$

(f) $L\{\cosh 4t + \cos 3t\} = L\{\cosh 4t\} + L\{\cos 3t\}$

$$= \left(\frac{s}{s^2 - 4^2}\right) + \left(\frac{s}{s^2 + 3^2}\right)$$

$$= \frac{s}{s^2 - 16} + \frac{s}{s^2 + 9}$$

(g) $L\{(3t-1)^2\} = L\{(3t-1)(3t-1)\}$

$$= L\{9t^2 - 6t + 1\}$$

$$= L\{9t^2\} + L\{-6t\} + L\{1\}$$

$$= 9\left(\frac{2!}{s^{2+1}}\right) + \left(\frac{-6}{s^2}\right) + \frac{1}{s}$$

$$= \frac{18}{s^3} - \frac{6}{s^2} + \frac{1}{s}$$

Quick Check 4.2

Find the Laplace Transforms of the following functions by using Table of Laplace Transforms:

1 $f(t) = -2 + e^{-2t} + t^4$

2 $f(t) = 3 + 5\sin 6t - 2e^{7t}$

3 $f(t) = \sin 5t + \cos 5t$

4 $f(t) = \sinh 7t + \cosh 4t$

5 $f(t) = e^{-2t}\cosh 3t$

6 $f(t) = -t + te^{-6t}$

7 $f(t) = 3t^5 + te^{2t}$

8 $f(t) = 7t + t^3 e^{-6t}$

9 $f(t) = t\sin 5t + t\cos 4t$

10 $f(t) = t(1 + t + t^2)$

11 $f(t) = (t+1)^3$

12 $f(t) = 2\sin 4t - 7\cos 3t$

13 $f(t) = \dfrac{e^{-2t}}{5} + 7 - e^{4t}\sinh 5t$

14 $f(t) = \dfrac{1}{3}\sin 5t + \sinh 7t$

Example 4.4

Special case (using trigonometry identity)

Find the Laplace Transforms of the following functions by using Table of Laplace Transforms.

(a) $f(t) = \sin^2 2t$ (b) $f(t) = \cos^2 2t$ (c) $f(t) = \sin t\cos t$ (d) $f(t) = (\sin t - \cos t)^2$

Solution

(a)
$$\begin{aligned} f(t) &= \sin^2 2t \\ &= \frac{1-\cos 4t}{2} \\ &= \frac{1}{2}L(1-\cos 4t) \\ F(s) &= \frac{1}{2}\left[L\{1\}-L\{\cos 4t\}\right] \\ &= \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4^2}\right] \\ &= \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+16}\right] \\ &= \frac{1}{2s}-\frac{s}{2\left(s^2+16\right)} \end{aligned}$$

$$\begin{aligned} \cos 2x &= 1-2\sin^2 x \\ 2\sin^2 x &= 1-\cos 2x \\ \sin^2 x &= \frac{1-\cos 2x}{2} \\ \sin^2 2x &= \frac{1-\cos 4x}{2} \end{aligned}$$

(b)
$$\begin{aligned} f(t) &= \cos^2 2t \\ &= \frac{\cos 4t+1}{2} \\ &= \frac{1}{2}(\cos 4t+1) \\ F(s) &= \frac{1}{2}\left[L\{\cos 4t\}+L\{1\}\right] \\ &= \frac{1}{2}\left[\frac{s}{s^2+4^2}+\frac{1}{s}\right] \\ &= \frac{1}{2}\left[\frac{s}{s^2+16}+\frac{1}{s}\right] \\ &= \frac{s}{2\left(s^2+16\right)}+\frac{1}{2s} \end{aligned}$$

$$\begin{aligned} \cos 2x &= 2\cos^2 x-1 \\ 2\cos^2 x &= \cos 2x+1 \\ \cos^2 x &= \frac{\cos 2x+1}{2} \\ \cos^2 2x &= \frac{\cos 4x+1}{2} \end{aligned}$$

(c)
$$\begin{aligned} f(t) &= \sin t\cos t \\ &= \frac{\sin 2t}{2} \\ F(s) &= \frac{1}{2}L\{\sin 2t\} \\ &= \frac{1}{2}\left(\frac{2}{s^2+2^2}\right) \\ &= \frac{1}{2}\left(\frac{2}{s^2+4}\right) \\ &= \frac{1}{s^2+4} \end{aligned}$$

$$\begin{aligned} \sin 2x &= 2\sin x\cos x \\ \frac{\sin 2x}{2} &= \sin x\cos x \end{aligned}$$

(d) $f(t) = (\sin t - \cos t)^2$

$$= (\sin t - \cos t)(\sin t - \cos t)$$
$$= \sin^2 t - \sin t \cos t - \cos t \sin t + \cos^2 t$$
$$= \sin^2 t - 2\sin t \cos t + \cos^2 t$$
$$= 1 - 2\sin t \cos t$$
$$= 1 - \sin 2t$$

$\sin^2 x + \cos^2 x = 1$

$\sin 2x = 2\sin x \cos x$

$$F(s) = L\{1\} - L\{\sin 2t\}$$
$$= \frac{1}{s} - \frac{2}{s^2 + 2^2}$$
$$= \frac{1}{s} - \frac{2}{s^2 + 4}$$

Theorem 2: First Shifting

We know that $f(t)$ is a function and $L\{f(t)\} = F(s)$ where a is a constant.

First Shifting theorem says that,

$$L\{e^{at} f(t)\} = F(s-a) \text{ where } s \longrightarrow (s-a)$$
$$L\{e^{-at} f(t)\} = F(s+a) \text{ where } s \longrightarrow (s+a)$$

Note: The notation $F(s-a)$ and $F(s+a)$ can be found when we replace $(s-a)$ or $(s+a)$ in $F(s)$ function.

Proof:

From the definition, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} . e^{at} . f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$
$$= \int_0^\infty e^{-pt} f(t)\,dt \text{ where } p = s - a$$
$$= F(p)$$
$$= F(s-a)$$

Example 4.5

By using First Shifting, find the Laplace Transform for $L\{e^{at}\}$.

Solution

$$f(t) = 1$$

$$L(1) = \frac{1}{s}$$

$$L\{e^{at}.1\} = \frac{1}{s-a}$$

Example 4.6

By using First Shifting, find the Laplace Transform for $L\{te^{at}\}$.

Solution

$$f(t) = t$$

$$L(t) = \frac{1}{s^2}$$

$$L\{e^{at}.t\} = \frac{1}{(s-a)^2}$$

Example 4.7

By using First Shifting, find the Laplace Transform for $L\{e^{at}\sin bt\}$.

Solution

$$f(t) = \sin bt$$

$$L(\sin bt) = \frac{b}{s^2+b^2}$$

$$L\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

Quick Check 4.3

Find the following Laplace Transforms by using First Shifting.

1 $L\left\{e^{2t}t^3\right\}$ **2** $L\left\{5t^3e^{-6t}\right\}$ **3** $L\left\{e^{-2t}\cosh 3t\right\}$

4 $L\left\{2e^{-3t}\right\}$ **5** $L\left\{e^{3t}\sinh 4t\right\}$ **6** $L\left\{e^{7t}\frac{4}{3}t^4\right\}$

Theorem 3: Multiplication with t^n

If $L\{f(t)\} = F(s)$, so

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}$$

$$= (-1)^n \frac{d^n}{ds^n}\{F(s)\} \text{ where } n = 1,2,3,..........$$

Proof:

By definition, $L\{f(t)\} = F(s) = \int_0^\infty e^{st} f(t)\, dt$.

Differentiate both sides with respect to s,

$$\frac{d}{ds} L\{f(t)\} = \frac{d}{ds} F(s)$$

$$= \int_0^\infty (-te^{-st}) f(t)\, dt$$

$$= -\int_0^\infty e^{-st} \{t.f(t)\}\, dt$$

$$= -L\{t.f(t)\}$$

Hence, $L\{t.f(t)\} = -\frac{d}{ds} F(s)$.

With the same process, $L\{t^2.f(t)\} = L[t\{t.f(t)\}]$

$$= -\frac{d}{ds} L[t.f(t)]$$

$$= -\frac{d}{ds}\left[-\frac{d}{ds} L\{f(t)\}\right]$$

$$= \frac{d^2}{ds^2} L[f(t)]$$

$$= \frac{d^2}{ds^2} F(s)$$

Example 4.8

By using multiplication by t^n, find the Laplace Transform for $L\{t\}$.

Solution

$$f(t) = 1$$

$$L(1) = \frac{1}{s}, \text{ where } n = 1$$

$$L(t) = (-1)^1 \frac{d^1}{ds^1}\left(\frac{1}{s}\right)$$

$$L(t) = \frac{1}{s^2}$$

Example 4.9

By using multiplication by t^n, find the Laplace Transform for $L\{t^2.e^{at}\}$.

Solution

$$f(t) = e^{at}$$

$$L(e^{at}) = \frac{1}{s-a}, \text{ where } n = 2$$

$$L\{t^2.e^{at}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-a}\right)$$

$$L(t^2 e^{at}) = \frac{2}{(s-a)^3}$$

Quick Check 4.4

Find the following Laplace transforms using multiplication by t^n.

1 $L\{3t\}$

2 $L\{t.e^{at}\}$

3 $f(t) = t^2 e^{-at}$

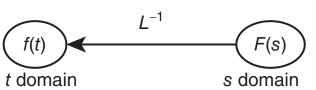
4 $f(t) = t \cos wt$

5 $f(t) = t \sin wt$

6 $f(t) = t^2 \cos at$

4.3 Understanding Inverse Laplace Transform

4.3.1 | Acquiring Inverse Laplace Transform for Simple Functions



It is a reverse process of Laplace Transform. If $L\{f(t)\} = F(s)$, then $f(t)$ is called inverse Laplace Transform of $F(s)$ and is written as $L^{-1}\{F(s)\} = f(t)$, where L^{-1} is the operator of inverse Laplace Transform.

Be aware that $L^{-1} \neq \frac{1}{L}$.

Example 4.10

Find the Inverse Laplace Transform for $\frac{4}{s}$.

Solution

$$L^{-1}\left\{\frac{4}{s}\right\} = 4$$

Example 4.11

Find the Inverse Laplace Transform for $\frac{6}{s^4}$.

Solution

$$L^{-1}\left\{\frac{6}{s^4}\right\} = 6L^{-1}\left\{\frac{1}{s^4}\right\}$$

Refer to the expression $\frac{1}{s^n}$, and identify the value of n.

$n = 4$,

so $\frac{t^{n-1}}{(n-1)!} = \frac{t^{4-1}}{(4-1)!} = \frac{t^3}{3!}$

$$\therefore\ 6L^{-1}\left\{\frac{1}{s^4}\right\} = 6\left(\frac{t^3}{3!}\right)$$
$$= t^3$$

Example 4.12

Find the Inverse Laplace Transform for $\frac{1}{s-6}$.

Solution

Referring to the expression $\frac{1}{s+a}$,

$$L^{-1}\left\{\frac{1}{s-6}\right\} = e^{6t}$$

Example 4.13

Find the Inverse Laplace Transform for $\frac{1}{\left(s+\frac{1}{2}\right)}$.

Solution

Referring to the expression $\frac{1}{s+a}$,

$$L^{-1}\left\{\frac{1}{\left(s+\frac{1}{2}\right)}\right\} = e^{-\frac{1}{2}t}$$

Example 4.14

Find the Inverse Laplace Transform for $\frac{2}{(s^2+4)}$.

Solution

Refer to the expression $\frac{\omega}{s^2+\omega^2}$, identify the value for ω; where $\omega = 2$.

So $L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin 2t$.

Example 4.15

Find the Inverse Laplace Transform for $\frac{3}{(9s^2+4)}$.

Solution

$$L^{-1}\left\{\frac{3}{9}\left(\frac{1}{s^2+\frac{4}{9}}\right)\right\} = L^{-1}\left\{\frac{1}{3}\left(\frac{1}{s^2+\left(\frac{2}{3}\right)^2}\right)\times\frac{\frac{2}{3}}{\frac{2}{3}}\right\}$$

$$= L^{-1}\left\{\frac{1}{2}\left(\frac{\frac{2}{3}}{s^2+\left(\frac{2}{3}\right)^2}\right)\right\}$$

$$= \frac{1}{2}\sin\frac{2}{3}t$$

Example 4.16

Find the Inverse Laplace Transform for $\frac{2s}{16s^2+9}$.

Solution

$$L^{-1}\left\{\frac{2}{16}\left(\frac{s}{s^2+\frac{9}{16}}\right)\right\} = L^{-1}\left\{\frac{1}{8}\left(\frac{s}{s^2+\left(\frac{3}{4}\right)^2}\right)\right\} = \frac{1}{8}\cos\frac{3}{4}t$$

Example 4.17

Find the Inverse Laplace Transform for $\frac{2}{9s^2-25}$.

Solution

$$L^{-1}\left\{\frac{2}{9}\left(\frac{1}{s^2-\frac{25}{9}}\right)\right\}=L^{-1}\left\{\frac{2}{9}\left(\frac{\frac{3}{5}\left(\frac{5}{3}\right)}{s^2-\left(\frac{5}{3}\right)^2}\right)\right\}=\frac{2}{15}\sinh\frac{5}{3}t$$

Quick Check 4.5

Find the Inverse Laplace Transforms for the following expressions:

1. $F(s)=\frac{3}{s^2-9}$
2. $F(s)=\frac{s}{s^2+16}$
3. $F(s)=\frac{s}{s^2-9}$
4. $F(s)=\frac{1}{s^5}$
5. $F(s)=\frac{12}{s^3}$
6. $F(s)=\frac{7s}{s^2+2}$
7. $F(s)=\frac{4}{s^2-9}$
8. $F(s)=\frac{5s}{s^2-4}$
9. $F(s)=\frac{15}{3s-4}$
10. $F(s)=\frac{s}{s^2+100}$
11. $F(s)=\frac{4s}{4s^2+49}$
12. $F(s)=\frac{4}{s^2-16}$
13. $F(s)=\frac{2}{s+5}$
14. $F(s)=\frac{10}{s^2+100}$
15. $F(s)=\frac{4s}{9s^2-25}$

Example 4.18

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{3s+8}{s^2+4}\right\}$ by using linearity.

Solution

$$L^{-1}\left\{\frac{3s+8}{s^2+4}\right\}=3L^{-1}\left\{\frac{s}{s^2+2^2}\right\}+4L^{-1}\left\{\frac{2}{s^2+2^2}\right\}=3\cos 2t+4\sin 2t$$

Example 4.19

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{4s+7}{s^2+9}\right\}$ by using linearity.

Solution

$$L^{-1}\left\{\frac{4s+7}{s^2+9}\right\}=4L^{-1}\left\{\frac{s}{s^2+3^2}\right\}+\frac{7}{3}L^{-1}\left\{\frac{3}{s^2+3^2}\right\}=4\cos 3t+\frac{7}{3}\sin 3t$$

Example 4.20

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{2s+5}{s^2-25}\right\}$ by using linearity.

Solution

$$L^{-1}\left\{\frac{2s+5}{s^2-25}\right\}=2L^{-1}\left\{\frac{s}{s^2-5^2}\right\}+L^{-1}\left\{\frac{5}{s^2-5^2}\right\}=2\cosh 5t+\sinh 5t$$

Quick Check 4.6

Find the Inverse Laplace Transforms for the following expressions by using linearity:

1. $\frac{s+3}{4(s^2+9)}$
2. $\frac{3s+5}{16s^2-9}$
3. $\frac{2s+\sqrt{3}}{3s^2+4}$

Example 4.21

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{1}{s-4}\right\}$ by using First Shifting.

Solution

$$L^{-1}\left\{\frac{1}{s-4}\right\}=e^{4t}L^{-1}\left\{\frac{1}{s}\right\}=e^{4t}(1)=e^{4t}$$

Example 4.22

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{3}{2s-6}\right\}$ by using First Shifting.

Solution

$$L^{-1}\left\{\frac{3}{2s-6}\right\}=L^{-1}\left\{\frac{3}{2(s-3)}\right\}=\frac{3}{2}L^{-1}\left\{\frac{1}{s-3}\right\}=\frac{3}{2}e^{3t}\left\{\frac{1}{s}\right\}=\frac{3}{2}e^{3t}$$

Example 4.23

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{5}{(s+2)^3}\right\}$ by using First Shifting.

Solution

$$L^{-1}\left\{\frac{5}{(s+2)^3}\right\}=5e^{-2t}L^{-1}\left\{\frac{1}{s^3}\right\}$$

By using formula $\frac{1}{5^n}$, identify value of n

$n=3$,

So, $\frac{t^{n-1}}{(n-1)!}=\frac{t^{3-1}}{(3-1)!}=\frac{t^2}{2!}=\frac{t^2}{2}$

$$L^{-1}\left\{\frac{5}{(2+2)^3}\right\}=5e^{-2t}\frac{t^2}{2}$$

$$=\frac{5}{2}e^{-2t}t^2$$

Example 4.24

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{3s}{(s-1)^4}\right\}$ by using First Shifting.

Solution

$$L^{-1}\left\{\frac{3s}{(s-1)^4}\right\}=3L^{-1}\left\{\frac{(s-1)+1}{(s-1)^4}\right\}=3L^{-1}\left\{\frac{(s-1)}{(s-1)^4}+\frac{1}{(s-1)^4}\right\}$$

$$=3L^{-1}\left\{\frac{1}{(s-1)^3}+\frac{1}{(s-1)^4}\right\}$$

$$=3e^{t}L^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^4}\right\}$$

$$=3e^{t}\left[L^{-1}\left\{\frac{1}{s^3}\right\}+L^{-1}\left\{\frac{1}{s^4}\right\}\right]$$

$$=3e^{t}\left[\frac{t^2}{2!}+\frac{t^3}{3!}\right]$$

$$=3e^{t}\left[\frac{t^2}{2}+\frac{t^3}{6}\right]$$

$$=\frac{3}{2}e^{t}t^2+\frac{1}{2}e^{t}t^3$$

Example 4.25

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{6}{(s-1)^2+4}\right\}$ by using First Shifting.

Solution

$$L^{-1}\left\{\frac{6}{(s-1)^2+4}\right\}=3L^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}=3e^{t}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}=3e^{t}\sin 2t$$

Example 4.26

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{s+1}{(s+3)^2+9}\right\}$ by using First Shifting.

Solution

$$\begin{aligned}L^{-1}\left\{\frac{s+1}{(s+3)^2+9}\right\}&=L^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+3^2}\right\}\\&=L^{-1}\left\{\frac{(s+3)}{(s+3)^2+3^2}-\frac{2}{(s+3)^2+3^2}\right\}\\&=e^{-3t}L^{-1}\left\{\frac{s}{s^2+3^2}-\frac{2}{s^2+3^2}\right\}\\&=e^{-3t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}\right\}-2L^{-1}\left\{\frac{1}{s^2+3^2}\times\frac{3}{3}\right\}\right]\\&=e^{-3t}\left(\cos 3t-\frac{2}{3}\sin 3t\right)\end{aligned}$$

Example 4.27

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{2s+3}{s^2-4s+20}\right\}$ by using First Shifting.

Solution

$s^2-4s+20 \quad ; \quad a=1, b=-4, c=20$

$$\left(s-\frac{4}{2}\right)^2+20-\left(-\frac{4}{2}\right)^2$$

$$(s-2)^2+16$$

Completing the square

$$\left(s\pm\frac{b}{2}\right)^2\pm c-\left(\frac{b}{2}\right)^2$$

$$\therefore \frac{2s+3}{s^2-4s+20}=\frac{2s+3}{(s-2)^2+16}=2L^{-1}\left\{\frac{s}{(s-2)^2+4^2}\right\}+3L^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$=2L^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+4^2}\right\}+3e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\times\frac{4}{4}\right\}$$

$$=2L^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\}+2L^{-1}\left\{\frac{2}{(s-2)^2+4^2}\right\}+3e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\times\frac{4}{4}\right\}$$

$$=2e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\}+4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\times\frac{4}{4}\right\}+\frac{3}{4}e^{2t}\sin 4t$$

$$=2e^{2t}\cos 4t+e^{2t}\sin 4t+\frac{3}{4}e^{2t}\sin 4t$$

$$=2e^{2t}\cos 4t+\frac{7}{4}e^{2t}\sin 4t$$

Quick Check 4.7

Find the Inverse Laplace Transforms for the following expressions by using First Shifting:

1 $\frac{2s}{(s+3)^2}$

2 $\frac{s}{(s-1)^3}$

3 $\frac{s+3}{s^2+6s+5}$

4 $\frac{8s}{s^2+4s+3}$

5 $\frac{2s-3}{2s^2-8s+16}$

6 $\frac{s-1}{2s^2+8s+11}$

4.3.2 Solving Inverse Laplace Transforms Using Partial Fraction Method

The function to be considered is of the form of a rational function of s, for example: $\frac{N(s)}{D(s)}$; where the degree of $D(s)$ is greater than the degree of $N(s)$.

The form of the partial fraction depends on the kind of factors in $D(s)$, which are real quadratic and linear.

Partial fraction is a sum of two or more proper fractions:

$$\frac{P(x)}{Q(x)}=\frac{a(x)}{b(x)}+\frac{c(x)}{d(x)}$$

The two simple fractions on the right-hand side are called the *partial fractions* of the expression on the left-hand side.

Types of Partial Fraction:

1. Denominator with a Single Linear Factor
2. Denominator with Repeated Linear Factors
3. Denominator with Quadratic Factors (can be factorized)
4. Denominator with Quadratic Factors (cannot be factorized)
5. Denominator with Repeated Quadratic Factors

Example 4.28

Find the Inverse Laplace Transforms for $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$ by using partial fractions.

Solution

Express as a sum of two fractions thus;

$$\frac{1}{(s+1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$$

$$\frac{1}{(s+1)(s-2)} = \frac{(A+B)s-2A+B}{(s+1)(s-2)}$$

$$\therefore A+B=0 \text{ and } -2A+B=1$$

Thus, $A = \frac{-1}{3}$ and $B = \frac{1}{3}$

Therefore, $L^{-1}\left\{\frac{-1}{3(s+1)} + \frac{1}{3(s-2)}\right\} = \frac{-1}{3}L^{-1}\left\{\frac{1}{(s+1)}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{(s-2)}\right\}$

$$= \frac{1}{3}\left(e^{2t} - e^{-t}\right)$$

Example 4.29

Find the Inverse Laplace Transform for $L^{-1}\left\{\frac{2s^2-3s+2}{(s+2)(s^2+4)}\right\}$ by using partial fraction.

Solution

Express as a sum of two fractions thus;

$$\frac{2s^2-3s+2}{(s+2)(s^2+4)} = \frac{A}{(s+2)} + \frac{Bs+C}{s^2+4}$$

$$\frac{2s^2-3s+2}{(s+2)(s^2+4)}=\frac{(A+B)s^2+Cs+4A+2C}{(s+2)(s^2+4)}$$
$$\therefore C=-3;\ A+B=2;\ 4A+2C=2$$

Thus, $A = 2$, $B = 0$ and $C = -3$.

Therefore, $L^{-1}\left\{\frac{2s^2-3s+2}{(s+2)(s^2+4)}\right\}=2L^{-1}\left\{\frac{1}{(s+2)}\right\}-3L^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$=2e^{-2t}-\frac{3}{2}\sin 2t$$

Quick Check 4.8

Find the following Inverse Laplace Transforms by using partial fractions:

1. $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$
2. $L^{-1}\left\{\frac{5s^3-6s-3}{s^3(s+1)^2}\right\}$
3. $L^{-1}\left\{\frac{1}{s^4-1}\right\}$
4. $L^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$
5. $L^{-1}\left\{\frac{3s^2+6}{(s-1)^2(s+2)}\right\}$
6. $L^{-1}\left\{\frac{3}{(s+2)(s^2+4s+5)}\right\}$

4.4 Defining Laplace Transform of Derivatives

4.4.1 | Explaining Differential Equations Using Laplace Transform

From the theorem,

$$\textit{If}\quad L\{y(t)\}=Y(s),\text{ then}$$
$$L\{y'(t)\}=sY(s)-y(0)$$
$$=sL\{y(t)\}-y(0)$$

Proof:
From the definition

$$L\{y'(t)\} = \int_0^\infty e^{-st} y'(t)dt$$
$$= \left[e^{-st} y(t)\right]_0^\infty + s\int_0^\infty e^{-st} y(t)dt$$
$$= -y(0) + sL\{y(t)\}$$
$$= sY(s) - y(0)$$

We can expand the theorem,

If $L\{y(t)\} = Y(s)$, then

$$L\{y'(t)\} = sY(s) - y(0)$$
$$L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$
$$L\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

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$$L\{y^n(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \dots\dots\dots - y^{n-1}(0)$$

4.4.2 | Solving Differential Equations Using Laplace Transform

Example 4.30

Solve the differential equation $y'' - 3y' - 10y = 2$ by using the Inverse Laplace Transform. Given that $y(0) = 1$ and $y'(0) = 2$

Solution

Step 1	Take the Laplace Transform for both sides. $L[y''(t)] - 3L[y'(t)] - 10L[y(t)] = L[2]$
Step 2	Substitute the theorem for LHS. Find the Laplace Transform for RHS. $\left[s^2Y(s) - sy(0) - y'(0)\right] - 3\left[sY(s) - y(0)\right] - 10\left[Y(s)\right] = \frac{2}{s}$

Step 3

Substitute the values of $y(0)=1$ and $y'(0)=2$.

$$\left[s^2Y(s)-s(1)-2\right]-3\left[sY(s)-1\right]-10\left[Y(s)\right]=\frac{2}{s}$$

Step 4

Solve for $Y(s)$.

$$s^2Y(s)-s-2-3sY(s)+3-10Y(s)=\frac{2}{s}$$

$$s^2Y(s)-3sY(s)-10Y(s)=\frac{2}{s}+s+2-3$$

$$Y(s)\left[s^2-3s-10\right]=\frac{2}{s}+\frac{s^2}{s}+\frac{2s}{s}-\frac{3s}{s}$$

Make the denominators equal

$$Y(s)\left[s^2-3s-10\right]=\frac{2+s^2-s}{s}$$

$$Y(s)=\frac{2+s^2-s}{s(s^2-3s-10)}$$

$$Y(s)=\frac{2+s^2-s}{s(s-5)(s+2)}$$

Factorize

Step 5

Solve for the inverse $y(t)=L^{-1}[Y(s)]$ using partial fraction.

$$y(t)=L^{-1}\left[\frac{2+s^2-s}{s(s-5)(s+2)}\right]$$

$$\frac{2+s^2-s}{s(s-5)(s+2)}=\frac{A}{s}+\frac{B}{s-5}+\frac{C}{s+2}$$

$$\therefore s^2-s+2=A(s-5)(s+2)+Bs(s+2)+Cs(s-5)$$

When $s=0$,

$$s^2-s+2=A(s-5)(s+2)$$
$$(0)^2-0+2=A(0-5)(0+2)$$
$$2=A(-5)(2)$$
$$2=-10A$$
$$A=-\frac{1}{5}$$

When $s=5$,

$$s^2-s+2=Bs(s+2)$$
$$(5)^2-5+2=B(5)(5+2)$$
$$22=35B$$
$$B=\frac{22}{35}$$

When $s=-2$,

$$s^2-s+2=Cs(s-5)$$
$$(-2)^2-(-2)+2=C(-2)(-2-5)$$
$$8=14C$$
$$C=\frac{8}{14}=\frac{4}{7}$$

$$L^{-1}\left[\frac{2+s^2-s}{s(s-5)(s+2)}\right]=L^{-1}\left[\frac{A}{s}+\frac{B}{s-5}+\frac{C}{s+2}\right]$$

$$=L^{-1}\left[-\frac{1}{5s}+\frac{22}{35(s-5)}+\frac{4}{7(s+2)}\right]$$

$$=-\frac{1}{5}L^{-1}\left[\frac{1}{s}\right]+\frac{22}{35}L^{-1}\left[\frac{1}{s-5}\right]+\frac{4}{7}L^{-1}\left[\frac{1}{s+2}\right]$$

$$=-\frac{1}{5}+\frac{22}{35}e^{5t}+\frac{4}{7}e^{-2t}$$

Quick Check 4.9

Solve the following differential equations by using the Inverse Laplace Transform.

1 $y'+y=\sin 3t$. Given $y(0)=0$.

2 $y''-y'-2y=e^{2t}$. Given $y(0)=0$ and $y'(0)=1$.

3 $y''-2y'+2y=e^{-t}$. Given $y(0)=0$ and $y'(0)=1$.

4 $y''+y=\cos 2t$. Given $y(0)=2$ and $y'(0)=1$.

SUMMARY

- Definition of Laplace Transform: $F(s)=L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$
- First Theorem of Laplace (Linearity): $L\{af_1(t)+bf_2(t)\}=aL\{f_1(t)\}+bL\{f_2(t)\}$
- Second Theorem of Laplace (First Shifting):

  $L\{e^{at}f(t)\}=F(s-a)$ where $s\to(s-a)$

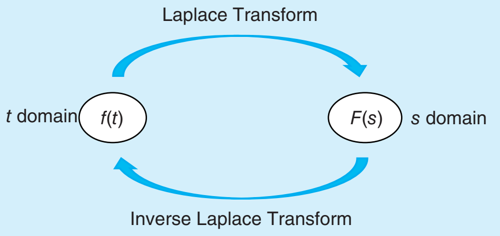
  $L\{e^{-at}f(t)\}=F(s+a)$ where $s\to(s+a)$
- Third Theorem of Laplace (Multiplication with t^n):

$$L\{t^nf(t)\}=(-1)^n\frac{d^n}{ds^n}L\{f(t)\}$$

$$=(-1)^n\frac{d^n}{ds^n}\{F(s)\}\text{ where } n=1,2,3,\ldots\ldots\ldots$$

- If $L\{f(t)\} = F(s)$, then the Inverse Laplace Transform of $F(s)$ is written as

$$L^{-1}\{F(s)\} = f(t)$$



Beware that $L^{-1} \neq \frac{1}{L}$

- Partial fraction is a sum of two or more proper fractions: $\frac{P(x)}{Q(x)} = \frac{a(x)}{b(x)} + \frac{c(x)}{d(x)}$
- Theorem of differential equation

$$\begin{aligned} &\text{If } L\{y(t)\} = Y(s), \text{then} \\ &L\{y'(t)\} = sY(s) - y(0) \\ &L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) \\ &L\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \end{aligned}$$

REVIEW QUESTIONS

1 Find the Laplace Transform for the function $f(t) = M + 7e^{4t}$ by using the Definition of Laplace.

2 Find the Laplace Transforms of the following functions by using Table of Laplace Transforms.

(a) $f(t) = e^{3t} + \cos 6t - e^{-2t}\cosh 3t$

(b) $f(t) = t\left(t^2+1\right)^2$

(c) $f(t) = 7e^{-3t} + \frac{3}{5} + 5t^3 + 6t$

(d) $f(t) = t^2 - 3t - 2e^{-t}\sin 3t$

(e) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

3 Find the Laplace Transform $L\left\{e^{-3t}\sin 2t\right\}$ by using First Shifting.

4 Find the Laplace Transform of the function $f(t) = t^2 \sin at$ using multiplication by t^n.

5 Find the Inverse Laplace Transforms for the following expressions:

(a) $F(s)=\dfrac{\sqrt{3}\,s}{3s^2-4}$ (b) $F(s)=\dfrac{9}{4s^4}+\dfrac{9}{s^2+100}$

(c) $F(s)=\dfrac{1}{2s-3}$ (d) $F(s)=\dfrac{3}{2s-6}$

(e) $F(s)=\dfrac{1}{s^2-25}$ (f) $F(s)=\dfrac{1}{4s^2+9}$

6 Find the Inverse Laplace Transform for the expression $\dfrac{3s-5}{4s^2+4s+1}$ by using First Shifting.

7 Find the following Inverse Laplace Transforms by using Partial Fraction:

(a) $L^{-1}\left\{\dfrac{s}{(s-1)(s^2-4)}\right\}$ (b) $L^{-1}\left\{\dfrac{s}{(s-1)(s+2)}\right\}$

(c) $L^{-1}\left\{\dfrac{11}{s^2-16}\right\}$ (d) $L^{-1}\left\{\dfrac{7s-2}{(s+4)(s^2+9)}\right\}$

(e) $L^{-1}\left\{\dfrac{5s-1}{s^2-s-2}\right\}$ (f) $L^{-1}\left\{\dfrac{13-s^2}{s(s^2+4s+13)}\right\}$

(g) $L^{-1}\left\{\dfrac{2s^2+11s-9}{s(s-1)(s+3)}\right\}$

8 Solve the following differential equations by using Inverse Laplace Transform.

(a) $y''+2y'+5y=0$. Given $y(0)=1$ and $y'(0)=0$

(b) $y'-2y=4$. Given $y(0)=3$

(c) $y'+y=5e^{2t}$. Given $y(0)=1$

(d) $y'+2y=12$. Given $y(0)=10$

(e) $y''+3y'+2y=24$. Given $y(0)=10$ and $y'(0)=0$